Graph-Constrained Surface Registration Based on Tutte Embedding

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Abstract

This work presents an efficient method to compute the registration between surfaces with consistent graph constraints based on Tutte graph embedding. Most natural objects have consistent anatomical structures, extracted as isomorphic feature graphs. For genus zero surfaces with ≥ 1 boundaries, the graphs are planar and usually 3-connected. By using Tutte embedding, each feature graph is embedded as a convex subdivision of a planar convex domain. Using the convex subdivision as constraint, surfaces are mapped onto convex subdivision domains and the registration is then computed over them. The computation is based on constrained harmonic maps to minimize the stretching energy, where curvy graph constraints become linear ones. This method is theoretically rigorous. The algorithm solves sparse linear systems and is computationally efficient and robust. The resulting mappings are proved to be unique and diffeomorphic. Experiments on various facial surface data demonstrate its efficiency and practicality.

1. Introduction

Surface registration is the fundamental tool for shape analysis and has been widely applied in computer vision and medical imaging fields. The primary challenge is how to obtain the precise dense diffeomorphic (one-to-one, onto, bijective) mapping between surfaces with large deformations. Natural physical deformation is usually nonrigid, for example, human facial surface deformation due to expression change or abnormality in various diseases such as autism. If precise registrations are given among surface frames, the deformation can be simulated and located for morphology understanding and quantified as similarity measurement for shape comparison, classification, recognition, and so on.

Motivation. Natural objects in real world have their own anatomical structures; objects in the same category have similar anatomy, such as human faces, hand palms, human bodies, and so on. Anatomical features can have various representation forms, such as the commonly used points and curves. Consistent anatomical feature constraints can serve as guidance in registration under different optimization criteria. A lot of research focuses on surface registration with feature point constraints [14, 15] and aim to generate diffeomorphisms [11, 20, 26, 3]. Curve constraints are usually discretized as point constraints [12]. Recently, the sophisticated geometric methods were introduced to exactly align curves in a complete and rigorous way [18, 27].

In practice, natural objects may have anatomical structure represented as a graph, which is embedded on the object surface with nodes and curvy edges, such as the feature graph on human facial surface (see Fig. 1). The nodes are the feature points (e.g., eye and mouth corners, nose tips), and the curvy edges are usually the landmark contours and curves (e.g., eye and mouth contours).

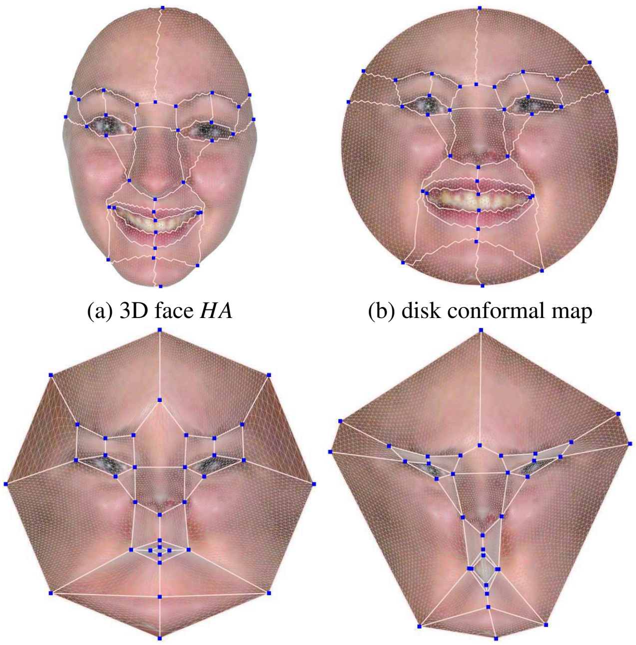
For surfaces with consistent feature graphs (isomorphic, with same nodes and connectivity), it is worthy of exploring the role of graphs in geometric registration, which will bring more benefits in practice by aligning both global topology and local geometry. The challenge is the "exact" alignment of curvy graphs while registering the interior (dense) surface areas with the diffeomorphism guarantee.

To our best knowledge, to date not much attention has been paid to tackle surface registration with graph constraints in a rigorous way. The goal of this work is:

To build dense diffeomorphic registration between surfaces with isomorphic graph constraints.

1.1. Approach Overview

This work presents an efficient method for tackling the problem of surface registration with graph constraints based on graph embedding. We call the feature graph embedded on the surface *decorative graph*, and the surface with graph on it *graph-decorated surface*. Surface without decorations is called *pure surface*. This work only considers the graph on genus zero surfaces with boundaries. These graphs are planar and extracted as 3-connected (i.e., each vertex connectivity ≥ 3) (see Fig. 1(a)). According to the Tutte embedding theorems [21, 4], the 3-connected planar graph on a surface can be embedded onto the Euclidean plane without crossing graph edges and every face is convex. Based



(a) 3D face *HA* (b) disk conformal map

(c) uniform convex map (d) weighted convex map

Figure 1. 3D surface mappings. (a) shows a happy (*HA*) facial surface decorated with a 3-connected feature graph. (b) shows a disk conformal mapping of the pure surface of (a). (c-d) show the convex mappings of (a), computed by harmonic map with uniform and weighted Tutte embeddings as constraints, respectively.

on the Radó theorem [17] and the main theorem 3.2 in this work, the harmonic map of the surface using the 2D convex subdivision as the constraint for the decorative graph is unique and diffeomorphic (see Fig. 1(c-d)). The generated mapping is called *convex mapping*. The strategy for registration is to compute the constrained harmonic map over 2D convex subdivision domains. The detail is as follows:

1) *Graph-driven surface mapping*. In detail, we first map the 3-connected planar graph to be a planar polygonal mesh, where each graph edge is mapped to a straight line segment by the uniform/weighted Tutte embedding. We then compute a harmonic map with the straight line graph constraint to map the whole surface onto the convex subdivision domain. The mapping result has convex faces with minimal stretches from the original surface, which defines a "canonical" shape representation for the graph-decorated surface.

2) *Graph-constrained surface registration*. By computing the graph-driven surface mappings for both source and target surfaces, the registration is converted to a mapping between two canonical domains. We compute a harmonic map with convex subdivision constraints to obtain the dense diffeomorphic registration between the surfaces to minimize the stretching energy.

This approach deals with surface as a whole and converts curvy graph edge constraints to linear ones. In contrast, conformal mapping (i.e., angle preserving mapping computed for pure surfaces) can also embed the feature graph onto a planar domain, but the graph edges are still curvy on that, as shown in Fig. 1(b). Thus canonical conformal maps can not be directly used for graph constrained registration.

The algorithms for both the Tutte embedding and the constrained harmonic map solve sparse linear systems, and therefore are efficient and robust, and are easy to implement. In addition, this framework can also work for genus zero surfaces with more than one boundaries, where the convex subdivision domain has convex inner holes. All the mappings are proved to be unique and diffeomorphic with exact alignment of graph constraints based on Theorem 3.2.

1.2. Previous Works

Constrained surface registration problem has been explored intensively in literature. Here we reviewed the most related ones. In terms of computational strategy, it is challenging to directly build surface registration in $\mathbb{R}^3$ space. Due to the fact that most natural surfaces to be registered are not rigid or isometric, the well-known iterative closest point (ICP) method and the Laplace spectral method won't work well. One efficient way is taking surface parameterization as an intermediate to apply Riemannian geometry tools and simplify computation. The LDDMM [1] method and the diffeomorphism geodesic [8] method use spherical parameterization for genus zero surfaces and compute the registration and deformation process simultaneously, and therefore the computation is highly nonlinear and time consuming.

Another category is canonical conformal map based methods which convert the registration problem between surfaces to that between 2D canonical domains. The 2D registration can be solved by feature point constrained harmonic mapping [12], quasiconformal mapping [25, 3], free-form deformation [7], intensity and curvature based energy optimization [13], or other image registration methods, such as Demon's method [19].

Besides the commonly used point constraints [14, 15, 20, 26], anatomical landmark curves have been used as feature constraints for geometric surface registration. Among the existing works, there are multiple approaches to handle these *curve constraints*: 1) a straightforward way is to convert them to point constraints, but this method cannot guarantee the alignment of curve intervals between points [12, 3]; 2) a method based on hyperbolic metric takes non-intersecting landmark curves as surface boundaries and makes them exactly aligned [18], but this method is highly nonlinear and changes the surface topology; and 3) a linear method converts curve constraints to straight line constraints [27], where canonical quasiconformal mapping for curve decorated surface were introduced, which maps the curves to horizontal and vertical straight lines on the 2D canonical domain. But this method requires the landmark curves reasonably distributed.

To our best knowledge, there is no work tackling graph constraints as a whole and using graph embedding in surface registration. Floater [4] has generalized the classical uniform Tutte graph embedding [21] to weighted graphs for triangular genus zero surface mesh parameterization and spline surface modeling [4]. There are some other works dealing with graph-based embeddings into the plane, e.g. using distances computed on a mesh and applying multidimensional scaling [28, 2]. In this work, we firstly employ the Tutte graph embedding to generate the "coarse" convex frames to drive the "dense" mapping between surfaces.

1.3. Contribution

The major contribution of this work is to present a strategy to solve the main problem, graph constrained surface registration, based on Tutte graph embedding. It extends the geometric mapping based registration framework to deal with graph-decorated surfaces. In detail,

1. It presents a diffeomorphic registration framework for genus zero surfaces with the exact alignment of 3-connected planar graph constraints. It can handle both simply-connected and multiply-connected domains.

2. It converts the graph constraints (both feature points and feature curves) to planar straight line graph constraints and exactly aligns them; and it takes the graph and the surface as a whole, linear and robust.

The proposed framework is rigorous with theoretical proof (see Section 3), and also practical with experimental and numerical verification (see Section 4). The experiments on a set of natural 3D facial surfaces with nonrigid deformations demonstrate the efficiency of the algorithms.

2. Mathematical Background

This section introduces the theoretic background used in this work. More details can be found in [16] for differential geometry, [17] for harmonic maps, and [10] for graph embedding.

2.1. Harmonic Map

Suppose a metric surface $(S,\mathbf{g})$ is a topological disk, a genus zero surface with a single boundary. By the Riemann mapping theorem, S can be conformally mapped onto the complex plane, $\mathbb{D}=\{z\in\mathbb{C}||z|<1\}$, $\phi:S\to\mathbb{D}$. ϕ is conformal implies $\mathbf{g}=e^{2\lambda(z)}dzd\bar{z}$, where λ is the area distortion factor, called the conformal factor. z is called an isothermal parameter of S, and ϕ is an isothermal parameterization.

Let $f:(\mathbb{D},|dz|^2)\to(\mathbb{D},|dw|^2)$ be a Lipschitz map between two disks, $z=x+iy$ and $w=u+iv$ are complex parameters. The *harmonic energy* of the map is defined as

$$E(f)=\int_{\mathbb{D}}(|w_z|^2+|w_{\bar{z}}|^2)dxdy. \qquad (1)$$

Definition 2.1 (Harmonic Map) *A critical point of the harmonic energy is called a harmonic map.*

If the mapping is harmonic, then it satisfies the Laplace equation $w_{z\bar{z}}=0$. In general, harmonic mapping is not necessarily diffeomorphic. If the restriction on the boundary is a homeomorphism, then the map from a topological disk to a planar disk is a diffeomorphism and unique.

Theorem 2.2 (Radó [17]) *Assume $\Omega\subset\mathbb{E}^2$ is a convex domain with a smooth boundary $\partial\Omega$ and a metric surface $(S,\mathbf{g})$ is a simply connected domain. Given any homeomorphism $\tau:\partial S\to\partial\Omega$, then the harmonic map $\phi:S\to\Omega$, such that $\phi=\tau$ on ∂S, is a diffeomorphism and unique.*

2.2. Graph Embedding

In graph theory, a graph G is k**-connected** if it requires at least k vertices to be removed to disconnect the graph, i.e., the vertex connectivity of G is **greater than or equal to** k **(**$>=k$**)**. A planar graph is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other. Such a drawing is called the planar embedding of the graph.

A 3-connected planar graph has special property that it has a planar crossing-free straight line embedding. W. T. Tutte (1963) gave a computational solution to this.

Definition 2.3 (Tutte Embedding [21]) *The Tutte embedding (or barycentric embedding) of a 3-connected planar graph is a crossing-free straight-line planar graph (polygonal mesh), where each interior vertex is at the average (or barycenter) of its neighboring positions. The outer face and the interior faces are convex.*

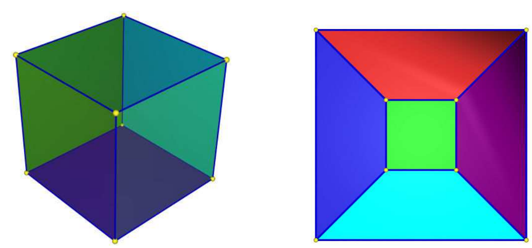


Figure 2. Tutte graph embedding for cube.

If the outer polygon is fixed, then the barycenter condition on the interior vertices determines their positions uniquely as the solution to a system of linear equations. Figure 2 shows an example. The resulting embedding is guaranteed to be unique, crossing-free and convex, by the Tutte's spring theorem [21].

Theorem 2.4 (Tutte's Spring Theorem [21]) *The solution to the linear equations is unique, and the embedding is* *always crossing-free. Specially, every face of the resulting planar embedding is convex.*

3. Computational Algorithms

The computation steps include: 1) Compute decorative graph embedding; 2) Compute harmonic map using graph embedding constraints; and 3) Compute dense registration by direct alignment or constrained harmonic map.

The surface is represented as a triangular mesh of genus zero and with a single boundary, denoted as $M = (V, E, F)$, where V, E, F represent vertex, edge and face set, respectively. The graph is 3-connected and planar, embedded on M, represented as $G = (V_G, E_G, F_G)$ on the mesh, where V_G, E_G, F_G represent graph node, edge and face set, respectively. Note that, each graph edge is embedded on the surface, and therefore is a curve, denoted as a chain of surface vertices. Thus, the surface M decorated with graph G is denoted as (M, G).

3.1. Isomorphic Graph Embedding

The goal of this step is to compute a straight line convex graph embedding of G, $\eta : G \to \hat{G}$ by the (generalized) Tutte embedding. We specify one graph face as the outer face, and place the graph nodes surrounding the outer face onto the unit circle uniformly. Then we compute the target positions for other nodes as the convex combination of neighboring nodes,

$$\{\hat{v}_i = \frac{\Sigma \bar{w}_{ij} \hat{v}_j}{\Sigma \bar{w}_{ij}}, \bar{w}_{ij} > 0\}, \forall v_i \in V_G.$$

In the classical Tutte embedding, the graph edge weight $\bar{w}_{ij}$ is set to be 1, corresponding to the barycentric coordinates. In order to respect the original geometry in some sense, we set $\bar{w}_{ij} = 1/|e[v_i, v_j])|$ [4], here $|e[v_i, v_j])|$ denotes the exact length of the curvy graph edge. We call these two cases as *uniform* and *weighted* convex graph embeddings, respectively. Solving these linear equations, we obtain all $\hat{v}_i$. The computational details and proofs can be found in [21, 4]. Accordingly, $\hat{G}$ defines a convex planar polygonal mesh Ω, which is a "convex subdivision" of a convex domain.

3.2. Graph-Driven Surface Mapping

The goal of this step is to compute a mapping $\phi : (M, G) \to (\Omega, \hat{G})$ (see Alg. 1). We use the obtained straight line convex graph embedding $\hat{G}$ as the hard constraints for the decorative graph G and map the whole surface onto a convex planar domain Ω by a constrained harmonic map.

In detail, it is to minimize the harmonic energy (the discrete version of (1))

$$E(\phi) = \Sigma_{[v_i, v_j] \in E} w_{ij} (\phi(v_i) - \phi(v_j))^2 \qquad (2)$$

under the convex subdivision constraint, formulated as

$$\begin{aligned} &\min E(\phi) \\ &s.t., \phi(l_k) = \hat{l}_k, \forall l_k \in G \cap M, \hat{l}_k = \eta(l_k), \end{aligned} \qquad (3)$$

where l_k is the curvy edge of the graph G, denoted as a chain of vertices on M, $\hat{l}_k$ is the target edge on the planar graph embedding $\hat{G}$, and w_{ij} denotes the edge weight (see Appendix). It means that the graph nodes are mapped to the exact positions in $\hat{G}$, the curvy graph edges are mapped to the straight line graph edges in $\hat{G}$. There is certain freedom that the vertices on graph edges can "slide" along the target straight line segments by the line relation.

The solution to (3) is equivalent to solving the linear system $\Delta \phi = 0$, where Δ is the Laplacian-Beltrami operator. It is discretized as the linear equations,

$$\Sigma_{[v_i, v_j] \in E} w_{ij} (\phi(v_i) - \phi(v_j)) = 0, \forall v_i \in V, \qquad (4)$$

with the linear constraints in (3). The algorithm has linear time complexity. The solution to target convex domain with convex subdivision achieves a unique diffeomorphism, guaranteed by the *Generalized Harmonic Map Theorem* 3.2. The proof is based on Lemma 3.1 on *convex combination map*. The details are given in Appendix.

Lemma 3.1 (Convex Combination [21, 6]) *Given a simply connected triangular mesh M and a convex domain Ω, if the map: $\phi : M \to \Omega$ is a convex combination map, i.e., for every interior vertex $v_i \in M$, $\phi(v_i) = \Sigma_{j=1}^{n} \lambda_{ij} \phi(v_j)$, where $\lambda_{ij} > 0$ and $\Sigma_j \lambda_{ij} = 1$, and if ϕ maps ∂M to $\partial \Omega$ homeomorphically, then ϕ is one-to-one.*

Theorem 3.2 (Harmonic Map to Convex Subdivision) *Assume $\Omega \subset \mathbb{R}^2$ is a convex domain with convex subdivision $\hat{G}$ and a smooth boundary $\partial\Omega$ and a metric surface $(M, \mathbf{g})$ is a simply connected domain with a decorative graph G. Given any homeomorphism between boundaries $\tau : \partial M \to \partial \Omega$, then the constrained harmonic map $\phi : (M, G) \to (\Omega, \hat{G})$, such that $\phi = \tau$ on ∂M and $\phi(G) = \hat{G}$, is a diffeomorphism and unique.*

This theorem can be straightforwardly extended to genus zero surfaces with multiply boundaries. The computational algorithm is the same. It implies that we can compute a constrained harmonic map to a convex domain with convex holes, which provides an approach to the problem of registration between multiply-connected domains.

3.3. Graph-Constrained Surface Registration

The goal is to find a diffeomorphism between two isomorphic graph-decorated surfaces such that the curvy graphs are exactly aligned (see Alg. 2). The main strategy is to employ the above graph embedding driven surface mappings to convert 2D/3D surfaces with irregular shaped

Algorithm 1 Graph-Driven Surface Mapping

Input: A triangular mesh with decorative graph (M,G)
Output: A triangular mesh with decorative graph $(\Omega,\hat{G})$
1: Compute Tutte embedding $\hat{G}$ of G
2: Compute harmonic map $\phi : (M,G) \to (\Omega,\hat{G})$ with constraints $\phi(G) = \hat{G}$ as (2-4)

decorative graphs to 2D straight line convex subdivision domains. Thus the desired registration can be efficiently obtained by minimizing harmonic energy (intuitively minimizing stretches) over the 2D domains. The solution is unique based on Theorem 3.2 and the computation is linear.

Registration framework. Suppose M_1, M_2 are the source and target surfaces to be registered, which are genus zero surfaces with boundaries, decorated with isomorphic 3-connected graphs G_1, G_2 as feature correspondence constraints, $G_1 \sim G_2$. The registration is defined as $f : (M_1,G_1) \to (M_2,G_2)$. *First*, we select the corresponding faces of G_1, G_2 as the outer faces and compute the planar graph embeddings $\hat{G}_k$ using Tutte graph embedding. *Second*, we compute the graph-driven surface mapping for (M_1,G_1) and (M_2,G_2), $\phi_k : (M_k,G_k) \to (\Omega_k,\hat{G}_k)$ via a constrained harmonic map using Alg. 1. *Finally*, we compute a constrained harmonic map $h : (\Omega_1,\hat{G}_1) \to (\Omega_2,\hat{G}_2)$ using Alg. 1 to minimize the harmonic energy $E(h)$ with the graph constraints $h(\hat{G}_1) = \hat{G}_2$, as defined in (3). Therefore, the registration $f = \phi_2^{-1} \circ h \circ \phi_1$, as shown in Diagram (5).

$$\begin{tikzcd} (M_1,G_1) \arrow[r, "f"] \arrow[d, "\phi_1"'] & (M_2,G_2) \arrow[d, "\phi_2"] \\ (\Omega_1,\hat{G}_1) \arrow[r, "h"] & (\Omega_2,\hat{G}_2) \end{tikzcd} \tag{5}$$

As an alternative approach, we can first compute the graph-driven mapping for the target surface, $\phi_2 : (M_2,G_2) \to (\Omega_2,\hat{G}_2)$, and then map the source surface to the target domain directly by computing a harmonic map, $\phi_1 : (M_1,G_1) \to (\Omega_1,\hat{G}_2)$, so that source graph nodes are exactly mapped to the target graph nodes and the source graph edges can slide along the target graph straight line edges.

Due to G_1 is isomorphic to G_2, if we employ the uniform Tutte embedding, then the embeddings are same, $\hat{G}_1 = \hat{G}_2$. The mapping h in (5) becomes identical, and can be computed by direct alignment of two 2D domains. This gives a simple and straightforward registration with isomorphic graph constraints. In order to improve the registration accuracy, we can add landmark point constraints on graph edges, and then compute a harmonic map by solving a similar system in (3), $\{h(p_{1,i}) = p_{2,i}\}, i = 1..n$, where $(p_{1,i}, q_{1,i})$ are point pairs. The resulting mapping is still a diffeomorphism, using the similar proof as Theorem 3.2.

Algorithm 2 Graph-Constrained Surface Registration

Input: Two triangular meshes with isomorphic decorative graphs $(M_k,G_k), k = 1,2$, $G_1 \sim G_2$
Output: A map $f : (M_1,G_1) \to (M_1,G_2)$ s.t. $f(G_1) = G_2$
1: Compute Tutte embedding $\hat{G}_k$ of G_k, $\eta : G_k \to \hat{G}_k$
2: Compute harmonic map $\phi_k : (M_k,G_k) \to (\Omega_k,\hat{G}_k)$
3: Compute harmonic map $h : (\Omega_1,\hat{G}_1) \to (\Omega_2,\hat{G}_2)$ s.t. $h(\hat{G}_1) = \hat{G}_2$
4: The resulting map is $f = \phi_1 \circ h \circ \phi_2^{-1}$

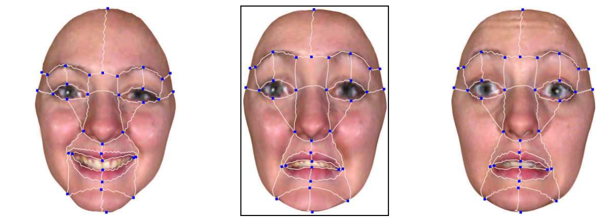
4. Experimental Results

We tested the proposed graph embedding-driven surface mapping and registration algorithms on various facial surface data. We utilize the anatomical feature graph constraints on facial surfaces and build a diffeomorphic registration framework for this type of graph-decorated facial surfaces. Experiments demonstrate that the proposed method gives a thorough shape representation for surfaces decorated with graphs considering both global and local structures, and is efficient and effective and therefore is promising for exploring dynamic morphometry in large-scale databases.

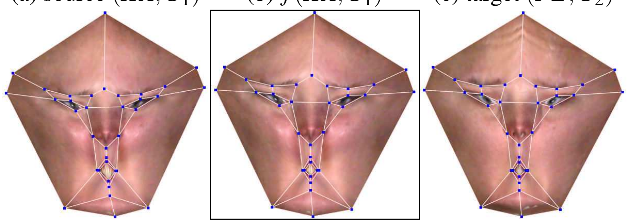
We tested the algorithms on a set of 3D human facial surfaces with expression change from private scans and public databases. The expression deformation is large, nonrigid and non-isometric.

Facial graph design and generation. For human facial surfaces, we employ the most prominent anatomical features including points, curves and contours around the eyes, mouth, nose and eye brows, and geometric features such as the symmetry axis and boundaries. These features can be either automatically computed or manually labeled (e.g., BU3FE [24]). We connect the feature points by shortest paths to construct the feature graph (see Fig. 1). There are various connecting patterns and the key is to generate a 3-connected graph. In our experiments, we refer to the natural muscle group of human facial expressions to divide the whole face. For facial surfaces with inner hole(s) (e.g., eye and mouth areas are inconsistent and need to be removed, such as open mouth/eye to closed one), we include the inner hole boundary in the feature graph (see Fig. 4).

Face mapping and registration. Figure 3 shows an example for faces from the same subject with different expressions. Our method can also work for genus zero surfaces with more than one boundaries. Figure 4 gives another example for faces with significantly different expressions. We remove the teeth area for the happy expression and slice the mouth open between lips for the sad expression. Then the surface is of genus zero with two boundaries. Figure 5 shows the registration accuracy by the consistent texture mapping results. In our registration, the source

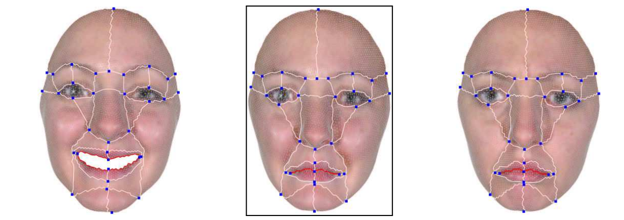


(a) source (HA, G_1) (b) $f(HA, G_1)$ (c) target (FE, G_2)

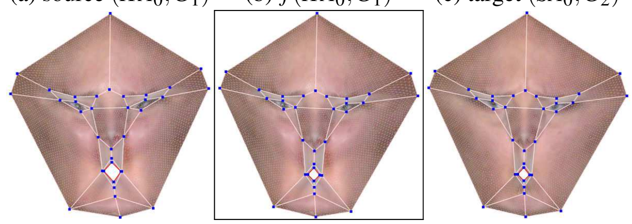


(d) convex $(\Omega_1, \hat{G}_1)$ (e) $h(\Omega_1, \hat{G}_1)$ (f) convex $(\Omega_2, \hat{G}_2)$

Figure 3. Facial surface registration between different expressions, HA (happy) and FE (fear), with feature graph constraints. The isomorphic feature graphs have 35 nodes, 61 edges, and 27 faces.



(a) source (HA_0, G_1) (b) $f(HA_0, G_1)$ (c) target (SA_0, G_2)



(d) convex $(\Omega_1, \hat{G}_1)$ (e) $h(\Omega_1, \hat{G}_1)$ (f) convex $(\Omega_2, \hat{G}_2)$

Figure 4. Facial surface registration between different expressions, HA_0 (happy) and SA_0 (sad), with feature graph constraints. The isomorphic feature graphs have 35 nodes, 61 edges, and 27 faces.

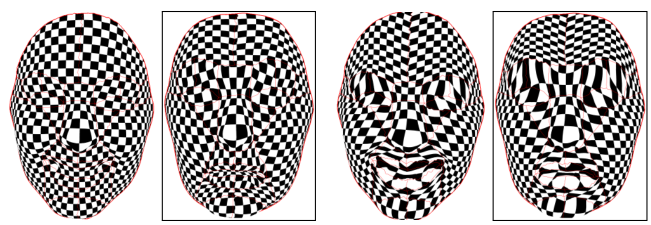
mesh is deformed onto the target. We use two types of texture coordinates for the source, one from the conformal mapping parameters and the other from the proposed weighted convex mapping parameters. Numerically, we compute the registration accuracy metric as $d(M_1, M_2) = \frac{1}{n}\Sigma_i^n \|\mathbf{r}(v_i) - \mathbf{r}(f(v_i))\|^2 + \|\mathbf{n}(v_i) - \mathbf{n}(f(v_i))\|^2$, where $\mathbf{r}$ denotes RGB function and $\mathbf{n}$ denotes normal function.

We tried the uniform and weighted convex graph embeddings (explained in Section 3.1) to guide the registration in our framework. The deformed source surfaces are very similar visually. We further used the above metric to test the difference. Then we have 1) uniform: $d(HA, FE) = 0.088825$, $d(HA_0, SA_0) = 0.100477$; 2) weighted: $d(HA, FE) = 0.077795$, $d(HA_0, SA_0) = 0.096933$. Details can be found in Table 1. It is obvious that the weighted graph embedding, which respects surface geometry, can achieve better registration accuracy.

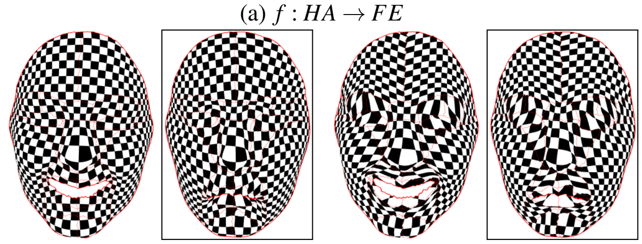
Method	$HA \to FE$
Uniform	0.019350 (RGB) + 0.069474 (Normal)
Weighted	0.016876 (RGB) + 0.060919 (Normal)
Method	$HA_0 \to SA_0$
Uniform	0.017790 (RGB) + 0.082687 (Normal)
Weighted	0.017060 (RGB) + 0.079874 (Normal)

Table 1. Comparison of registration accuracy.

We have tested 30 pairs of facial surface registration among the BU3DFE faces (about 14k triangles). The averaged running time is within 10 seconds. All experiments show that our method can handle facial surfaces with large natural deformations and generate convincing registration results. It has great potential in large-scale facial recognition and expression analysis tasks.



(a) $f: HA \to FE$



(b) $f: HA_0 \to SA_0$

Figure 5. Visualization of registration by texture mappings. *Left* pair uses coordinates of disk conformal mapping of source; *Right* pair uses coordinates of weighted convex mapping of source.

4.1. Comparison

Comparison to other options of convex mapping. With different boundary conditions, the pure surfaces (without considering decorative graphs) can be mapped to different canonical domains, such as by disk conformal mapping and convex harmonic mapping. The feature graph becomes irregular planar graphs with curvy edges on such pure surface mapping result, as shown in Fig. 1(b). Intuitively, these maps can define a planar straight line graph embedding by simply connecting the nodes on the planar domain, but cannot guarantee crossing-free (with self-flipping) property and may generate concave faces, and skinny faces (not perceptively pleasing). Therefore there is no guarantee of guiding a diffeomorphic surface mapping using such straight line graphs. Our method based on Tutte embedding can conquer all the above limitations and has theoretical guarantee.

Comparison to patch-to-patch registration. There is no existing method to handle the graph constraints as a whole in surface registration. An intuitive and straightforward approach is to register the corresponding patches separately and then glue all the patches along curvy graph edges. However, this method requires special handling (e.g., set target position based on length ratio) to generate one-to-one mapping at the graph edges, and has no guarantee to minimize the global stretches. In contrast, our registration makes the vertices on the source graph edge slide on the corresponding target graph edge to minimize the stretching energy, generates a diffeomorphic mapping for the whole domain including the graph edges; and is computed as a whole and therefore more efficient. We compared the harmonic energies induced by the two registrations from face HA to face FE: 1) the patch-to-patch mapping: 11.119, and 2) the convex mapping, 10.960 (smaller), which are computed by Eqn. (2) with mean value coordinates as edge weights.

4.2. Discussion

This work gives an efficient and simple method to register graph-decorated surfaces (genus zero and with boundaries) based on planar graph embedding, which, intuitively, offers straight line frames as coarse guidance for dense surface registration.

Rigor and optimality. The proposed method has solid theoretical background and guarantees the uniqueness and diffeomorphism. The Jacobian determinants of all the vertices are positive ($J > 0$) in all the resulting mappings, which verifies the diffeomorphism property in practice. The registration is computed by constrained harmonic map and therefore is optimal in terms of stretching energy.

Novelty. This work rigorously solves the problem of diffeomorphic dense surface registration with graph constraints, and presents the exact alignment of decorative graphs through Tutte embedding by making full use of its convex subdivision property. Furthermore, this framework can handle the diffeomorphic registration of multiply-connected domains by treating the inner boundaries as convex faces in the embeddings.

Efficiency and practicality. Both the Tutte embedding and constrained harmonic map solve sparse linear systems. The algorithm is stable and robust, which has been verified from a large amount of tests. Therefore, the method is practical.

Generality and other possibilities. In practice, possible preprocessing steps include hole-filling and boundary consistency check can be applied to the surfaces to be registered. For objects without explicit graph features, some auxiliary feature curves should be created (e.g., by shortest paths). The accuracy of graph constraints affect the accuracy of registration. The selection of the outer face generates different shapes and resolutions for the regions in the convex mapping, therefore, affects the accuracy slightly. In addition, the presented method minimizes harmonic energy under constraints; the resulted map is as smooth as possible. With the map, more optimality criteria such as angle or area distortion minimization [9, 26, 23, 22] can be integrated. Other sophisticated graph embeddings can also be introduced to this framework for more complicated graph-decorated surfaces. We will investigate the above in our extension work.

Potential to biomedical applications. In practice, some natural objects are associated with landmark graphs, such as human body, hands, faces, brains, and medical images. The proposed framework is general and extensible for diverse registration applications. Through registration, morphometry analysis can be performed in medical databases for diagnosis, such as facial surface registration for autism disease and cortical surface registration for Alzheimer's disease. Note that anatomical graphs on cortical surfaces (not like facial feature graphs) may not be consistent, so additional processing on graphs is required.

5. Conclusion and Future Work

This work presents an efficient surface registration framework specially considering isomorphic *decorative graph* constraints, and applies that for facial surface registration with anatomical constraints. The main strategy is to generate convex mappings which convert curvy feature graphs to straight-line convex subdivisions based on Tutte embedding, so that surfaces can be registered over convex subdivision domains, and the resulted mappings are guaranteed to be unique and diffeomorphic. The framework is theoretically rigorous, efficient and practical. Furthermore, it can be applied to biomedical problems where feature graphs are associated, such as facial surfaces for autism study and cortical surfaces for Alzheimer's study, which will be explored in our future work.

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Appendix

To prove Theorem 3.2, based on Lemma 3.1, we only need to prove that there exists a design of graph constraints in computing harmonic map over convex subdivision domains such that each vertex has a convex combination.

Proof: In the computation of the harmonic map with convex subdivision constraints, the target domain is convex and the interior vertices (except vertices on the graph) are convex combinations of their one-ring neighboring vertices. If the standard cotangent weights are employed, then the weights can be positive everywhere by refining the mesh triangulation; if the mean value coordinates in [5] are used, then the weights are naturally guaranteed to be positive. Both kinds of weights are used to approximate harmonic map. In our current computation, we used mean value coordinates.

Based on Lemma 3.1, if there exists a convex combination for each graph node, then the mapping is a diffeomorphism. We show this as follows. As the subdivision derived by graph embedding is convex, i.e., no line through the interior graph node can make all the neighboring vertices to one side, each interior graph node must lie inside the convex hull of its neighbors on adjacent graph edges. For the vertex on the graph edge, we limit it as the convex combination of the two neighboring edges on the same graph edge; that means the vertex can only slide on the graph edge. Thus, we proved that there exists a convex combination for every vertex and therefore the map is diffeomorphic and unique. □

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