The Common Self-polar Triangle of Concentric Circles and Its Application to Camera Calibration

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In projective geometry, the common self-polar triangle has often been used to discuss the position relationship of two planar conics [3]. However, there are few researches on the properties of the common self-polar triangle, especially when the two planar conics are special conics. In this paper, we explore the properties of the common self-polar triangle, when the two conics happen to be concentric circles. The main contribution of this paper is that we initiate a new perspective to look into circle based camera calibration problem. We believe that other calibration methods using different circle patterns can benefit from this perspective, especially for the patterns which involve more than two circles.

The application in this paper is not a complex problem, our purpose is not to outperform other calibration methods [2, 4, 5], but to develop some new theories, which would be easy to follow, implement and may be potentially useful in other fields of computer vision.

Given two concentric circles, we obtain some important results below:

Result 1. Two concentric circles have infinite many common self-polar triangles.

Result 2. All common self-polar triangles of two concentric circles share one common vertex and the opposite side of this vertex lies on the same line, which are the circle center and the line at infinity of the support plane.

Result 3. All common self polar triangles of two concentric circles are right triangles.

We show in this paper how to recover the vertices of the common self-polar triangle. Let point $\mathbf{x}$ and line p are the common pole-polar of $\mathbf{C}_1$ and $\mathbf{C}_2$. The following relationship should be satisfied:

$$\begin{aligned} p &= \mathbf{C}_1\mathbf{x} \\ p &= \lambda\mathbf{C}_2\mathbf{x}, \end{aligned} \qquad (1)$$

where λ is a scalar parameter. Subtracting the equations in (1), we get $(\mathbf{C}_1 - \lambda\mathbf{C}_2)x = 0$. By multiplying the inverse of $\mathbf{C}_2$ on both sides, we obtain the following equation:

$$(\mathbf{C}_2{}^{-1}\mathbf{C}_1 - \lambda\mathbf{I})\mathbf{x} = 0. \qquad (2)$$

From the equation of (2), we find the common poles for $\mathbf{C}_1$ and $\mathbf{C}_2$ are the eigenvectors of $\mathbf{C}_2{}^{-1}\mathbf{C}_1$. If $\mathbf{C}_1$ and $\mathbf{C}_2$ are two concentric circles, we find that:

Result 4. Given two concentric circles $\mathbf{C}_1$ and $\mathbf{C}_2$, the matrix $\mathbf{C}_2{}^{-1}\mathbf{C}_1$ has three eigenvalues, of which two are identical and one is different. The corresponding eigenvectors of the identical eigenvalues determine the line at infinity. The corresponding eigenvector of the different eigenvalue is the circle center.

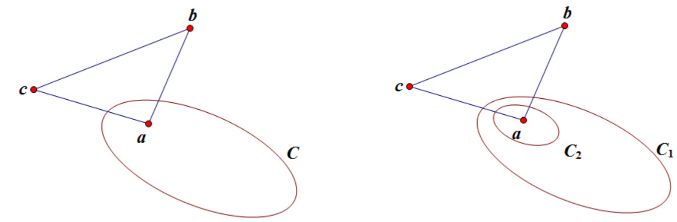
Based on those results and the fact pole-polar relationship is preserved under projective transformation [1], we establish an algorithm to recover the imaged circle center and the vanishing line. The recovery algorithm includes the following steps:

Step 1: Extract two concentric circles images $\widetilde{\mathbf{C}_1}$ and $\widetilde{\mathbf{C}_2}$.
Step 2: Compute $(\lambda, \mathbf{x})$ of $\widetilde{\mathbf{C}_1}$ and $\widetilde{\mathbf{C}_2}$. Let three eigenpairs be $(\lambda_1, \mathbf{x_1})$, $(\lambda_2, \mathbf{x_2})$, and $(\lambda_3, \mathbf{x_3})$, in which the values of λ_2 and λ_3 are identical.
Step 3: Calculate the cross product of $\mathbf{x_2}$ and $\mathbf{x_3}$ and let it be v. The imaged circle center is $\mathbf{x_1}$ and the vanishing line is v.

Once the vanishing line is recovered, two calibration algorithms can be obtained.

The first algorithm is given as follows:

This is an extended abstract. The full paper is available at the Computer Vision Foundation webpage.



(a) (b)

Figure 1: (a) $\triangle$**abc** is a self-polar triangle with respect to conic **C** when polars of a, b and c are lines bc, ac and ab, respectively. (b) $\triangle$**abc** is the common self-polar triangle of two disjoint conics $\mathbf{C}_1$ and $\mathbf{C}_2$ when $\triangle$**abc** is a self-polar triangle with respect to both $\mathbf{C}_1$ and $\mathbf{C}_2$.

Step 1: Extract the images of two concentric circles $\widetilde{\mathbf{C}_1}$ and $\widetilde{\mathbf{C}_2}$.
Step 2: Calculate the eigenvectors of $\widetilde{\mathbf{C}_2}^{-\mathbf{1}}\widetilde{\mathbf{C}_1}$. Then, recover the vanishing line.
Step 3: Find the imaged circular points by intersecting the circle image with the vanishing line.
Step 4: For three views, repeat the above steps three times.
Step 5: Determine IAC by using imaged circular points and obtain K using the Cholesky factorization.

The second algorithm is given as follows:
Step 1: Extract the images of two concentric circles $\widetilde{\mathbf{C}_1}$ and $\widetilde{\mathbf{C}_2}$.
Step 2: Calculate the eigenvectors of $\widetilde{\mathbf{C}_2}^{-\mathbf{1}}\widetilde{\mathbf{C}_1}$. Then, recover the imaged circle center ant the vanishing line.
Step 3: Randomly form two common self-polar triangles and calculate the conjugate pairs.
Step 4: For three views, repeat the above steps three times.
Step 5: Determine IAC by using orthogonality and obtain K using the Cholesky factorization.

All theories in this paper are clearly derived and accurate results are achieved in synthetic and real experiments. Our conclusion is that calibration methods derived from the properties of the common self-polar triangle of concentric circles is easy to follow and implement.

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