UniHIST: A Unified Framework for Image Restoration With Marginal Histogram Constraints

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A histogram is a discrete approximation of the probability distribution of a continuous random variable. For high dimensional signals such as images, when one treats individual data points as independent samples from a random source, the corresponding histogram is known as a *marginal histogram* [2]. Most vision applications interpret a marginal histogram as an approximate description of the underlying distribution, specifying the probability of the data points taking a particular value. However, a marginal histogram collected from an image provides more information as it represents an *ensemble* constraint on the image, specifying the proportion of the pixels taking each value. Several recent studies have shown that image restoration tasks can benefit from such ensemble constraints, but incorporating these constraints with existing restoration methods in a numerically stable manner remains a challenging problem [1].

In this work, we emphasize the aspect of marginal histograms as ensemble constraints and introduce a unified framework, UniHIST, to incorporate such constraints in image restoration problems. In UniHIST, we use the *quadratic Wasserstein* (W_2) distance [4] to measure the statistical distance between the marginal histograms of the output image and the reference histogram. The W_2 distance can be computed directly from data without relying on density estimation, providing a smooth and differentiable form to measure the dissimilarity between histograms. By including this term in an optimization framework, UniHIST can readily work with most existing restoration methods. We demonstrate the benefits of marginal histograms and the effectiveness of UniHIST through two applications: denoising of pattern images and non-blind deconvolution of natural images. We show that UniHIST enhanced restoration algorithms lead to improved restoration quality over existing state-of-the-art methods.

Method The W_2 distance between two probability measures p,q over the real line is defined as the variational solution to the Monge problem [4]:

$$W_2^2(p,q) = \inf_{\phi} \int_{-\infty}^{\infty} (x-\phi(x))^2 p(x) dx \tag{1}$$

where the infimum is over all deterministic functions $\phi : \mathcal{R} \mapsto \mathcal{R}$ that transfer an arbitrary random variable x with distribution p to a new variable $\phi(x)$ with distribution q. A fundamental result in the optimal transport theory is that the optimal ϕ minimizing Eq.(1) has a closed-form solution:

$$\phi_{p\to q}(x) = F_q^{-1}\left(F_p(x)\right). \tag{2}$$

where F_p is the *cumulative distribution function* of distribution p, and F_q^{-1} is the *percentile function* of distribution q.

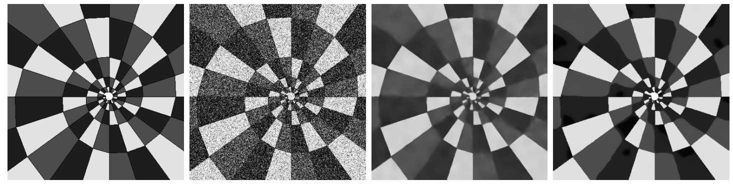
For our framework, we need to measure the statistical distance between the marginal histogram $h_{\mathbf{x}}$ of an image $\mathbf{x}$ with n pixels and a given marginal histogram h_q. By treating $\mathbf{x} = (x_1, \cdots, x_n)^T$ as n independent samples drawn from a distribution p, and h_q as the discrete approximation of another distribution q, we introduce an empirical $\hat{W}_2$ measure as follows:

$$\hat{W}_2^2(p,q) = \hat{W}_2^2(h_{\mathbf{x}}, h_q) = \min_{\phi} \frac{1}{n} \sum_{i=1}^{n} (x_i - \xi_i)^2 \tag{3}$$

where function ϕ maps x_i to $\xi_i = \phi(x_i)$, such that the transformed samples $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_n)^T$ satisfy the marginal histogram h_q. Analogous to the continuous case, the optimal $\hat{\phi}$ for Eq.(3) is given as [4]:

$$\xi_i = \hat{\phi}_{h_{\mathbf{x}}\to h_q}(x_i) = F_{h_q}^{-1}(F_{h_{\mathbf{x}}}(x_i)) \tag{4}$$

where $F_{h_q}^{-1}$ and $F_{h_{\mathbf{x}}}$ are the percentile function and the cumulative distribution function constructed from h_q and $h_{\mathbf{x}}$ respectively. A notable property of the empirical W_2 measure is that it can be evaluated in the quadratic form of $\mathbf{x}$, which allows easy integration with existing image restoration methods.



(a) Original Image (b) Noisy Observation (c) NLDD [3] (d) NLDD+UniHIST

Figure 1: A visual example for pattern image denoising. The original image (a) is degraded with a strong Gaussian noise ($\sigma^2 = 0.1$). (c) PSNR = 20.29, SSIM = 0.83 (d) **PSNR = 24.80**, **SSIM = 0.90**. By enforcing marginal intensity histogram constraints in the denoising process, our method (NLDD+UniHIST) significantly improves the visual quality and the quantitative results over the original NLDD method.

Suppose we have knowledge about the clean image $\mathbf{x}$, specifying the marginal histogram of $\mathbf{x}$ in a linearly transformed domain such as gradients or wavelet coefficients. We would like to use this information in the restoration method. We cast the problem into the following form:

$$\min_{\mathbf{x}} \frac{\lambda}{2}\|\mathbf{y} - A\mathbf{x}\|_2^2 + \Gamma(\mathbf{x}) + \frac{\beta}{2}\hat{W}_2^2(h_{B\mathbf{x}}, h_r) \tag{5}$$

where the first term represents the ℓ_2 difference between the observation $\mathbf{y}$ and the image formation model $A\mathbf{x}$, Γ represents a regularization term on $\mathbf{x}$, B is a linear transform, and $\hat{W}_2^2$ measures the discrepancy between the marginal histograms of $B\mathbf{x}$ and a reference histogram h_r in the transformed domain. Following the definition in (3), we introduce an auxiliary vector $\boldsymbol{\xi}$ (of the same dimension as $B\mathbf{x}$) and formulate UniHIST as follows:

$$\min_{\mathbf{x},\boldsymbol{\xi}} \frac{\lambda}{2}\|\mathbf{y} - A\mathbf{x}\|_2^2 + \Gamma(\mathbf{x}) + \frac{\beta}{2}\|\boldsymbol{\xi} - B\mathbf{x}\|_2^2 \quad \text{s.t.}\, h_{\boldsymbol{\xi}} = h_r \tag{6}$$

This problem can be readily solved with block coordinate descent by alternatively minimizing Eq.(6) with regards to $\boldsymbol{\xi}$ and $\mathbf{x}$ respectively. The $\mathbf{x}$ subproblem comes with two quadratic terms and a regularization term Γ on $\mathbf{x}$, which can be efficiently solved through half-quadratic splitting or proximal algorithms; the $\boldsymbol{\xi}$ subproblem can be solved in closed-form using Eq.(4).

Applications We provide two applications of the proposed UniHIST framework: denoising of pattern images and non-blind deconvolution of natural images. For pattern images, we show that by enforcing marginal intensity histogram constraints, UniHIST significantly improves the visual quality and the quantitative results over the state-of-the-art denoising methods; for natural images, we show that by including marginal gradient histogram constraints, UniHIST enhances the mid-level visual details for non-blind deconvolution algorithms. Figure 1 presents one visual example for pattern image denoising. Please refer to the paper and the supplemental material for further details.

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This is an extended abstract. The full paper is available at the Computer Vision Foundation webpage.