On the Minimal Problems of Low-Rank Matrix Factorization

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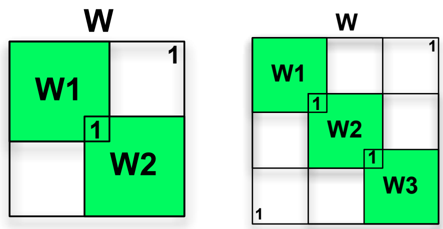

Figure 1: Examples of rank-2 constructive extensions. Several smaller minimal cases are glued together with a few extra constraints. For these extensions the multiplicity of solutions grows as $n_W = n_{W_1} n_{W_2}$ respectively $n_W = 2 n_{W_1} n_{W_2} n_{W_3}$. We have implemented the method of extending the algorithms, i.e. $\{f_{W_1}, f_{W_2}, \ldots f_{W_n}\} \to f_W$.

Low-rank matrix factorization is an essential problem in many areas including computer vision, with applications in e.g. affine structure-from-motion, photometric stereo, and non-rigid structure from motion. However, very little attention has been drawn to minimal cases for this problem or to using the minimal configuration of observations to find the solution. Minimal problems are useful when either outliers are present or the observation matrix is sparse. In this paper, we first give some theoretical insights on how to generate all the minimal problems of a given size using Laman graph theory [1, 2]. We then propose a new parametrization and a building-block scheme to solve these minimal problems by extending the solution from a small sized minimal problem.

Low rank matrix factorization is usually formulated as minimizing

$$\min_{U,V} \|W \odot (X - U^T V)\|, \tag{1}$$

where $\|.\|$ is a matrix norm – typically the L_1- or L_2-norm. The binary matrix $W \in \{0,1\}^{m \times n}$ is used to indicate if a certain entry $X(i,j)$ is present ($W(i,j) = 1$) or missing ($W(i,j) = 0$). Here $\odot$ is the Hadamard product, that is, the element-wise product.

An index matrix W is said to be *rigid* if for general data, the low-rank matrix factorization problem given by

$$W \odot (X - U^{\mathrm{T}} V) = 0, \tag{2}$$

is *locally well defined*. A minimal low-rank matrix factorization problem is finding two factor matrices U and V that exactly solves (2), where W is a minimal index matrix and X is the measurement matrix. For the minimal problem, with general coefficients, characterized by a minimal index matrix W, there is a finite number $n_W > 0$ of solutions, where n_W only depends on the index matrix W.

The minimal problems in low-rank matrix factorization can be constructed in a recursive way. The idea is that one starts with the smallest index matrix, and by a series of extensions every index matrix can be generated. For example, for $r = 2$, the smallest index matrix is

$$W = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \tag{3}$$

From this we can construct new minimal index matrices. We distinguish between *constructive* extensions and *non-constructive* extensions. For a constructive extension from W to W', one can infer the number of solutions $n_{W'}$ from n_W and construct the solver, denoted by $f_{W'}$ from f_W. For non-constructive extensions, it can be shown that W is minimal if and only if W' is minimal. However, we can in general neither infer the number of solutions $n_{W'}$ from n_W nor derive a solver $f_{W'}$ from f_W. We propose extensions and reductions which are denoted *Henneberg-k extensions/reductions*. Of these Henneberg-1 is constructive, whereas Henneberg-k are generally non-constructive. We also show how several minimal problems can be "glued" in a constructive way, as in Fig. 1.

The minimal solvers can be used in a RANSAC framework to handle both missing data and outliers. We have tested our methods on both synthetic data and real data in two applications, namely affine structure from motion and non-rigid structure-from-motion. We have compared our method to a number of the state-of-the-art methods. In Table 1 the results on non-rigid structure-from-motion are shown. The visual result of our method is shown in Fig. 2.

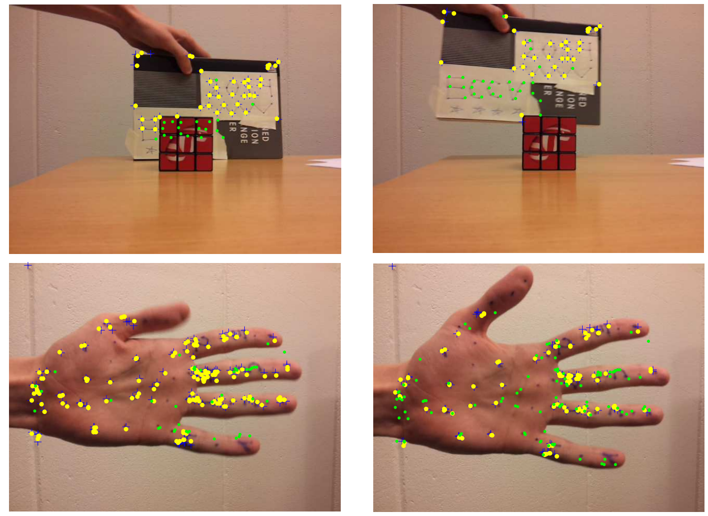

Figure 2: The results on the *Book* (top) and *Hand* (bottom) dataset using our method with an extra 10% missing data added in the blocks. The blue crosses are the measurements. The yellow dots are the recovered measurements (usually coinciding with the actual measurement). The green dots are the recovered missing data.

	Algorithm		10 % missing data	
Dataset	[3]	*Our*	[3]	*Our*
Book	0.3522	**0.1740**	8.0436	**0.1772**
Hand	0.8613	**0.6891**	1.5495	**0.7297**

Table 1: The result on linear shape basis estimation on the *Book* and *Hand* dataset, where the second experiment contains an extra 10% missing data. The numbers depict the Frobenius errors using our method compared to the method of [3].

[1] Lebrecht Henneberg. *Die graphische Statik der starren Systeme*, volume 31. BG Teubner, 1911.

[2] Gerard Laman. On graphs and rigidity of plane skeletal structures. *Journal of Engineering mathematics*, 4(4):331–340, 1970.

[3] Viktor Larsson, Carl Olsson, Erik Bylow, and Fredrik Kahl. Rank minimization with structured data patterns. In *ECCV*, 2014.

This is an extended abstract. The full paper is available at the Computer Vision Foundation webpage.