Bilinear Random Projections for Locality-Sensitive Binary Codes

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Locality-sensitive hashing (LSH) [1, 3, 5] is not only known as a data-independent indexing method for approximate similarity search, but also a compression algorithm for large-scale learning problems, where randomly generates binary codes such that two similar items in database are nearly embedded into Hamming space. Different similarity metric leads to various LSH, including angle preservation [1], ℓ_p norm ($p \in (0,2]$) [2], and shift-invariant kernels [5].

Most of high-dimensional visual descriptors for images exhibit a natural matrix structure. When visual descriptors are represented by high-dimensional feature vectors and long binary codes are assigned, a random projection matrix requires expensive complexities in both space and time. Recently, bilinear projections are adopted to the angle-preserving LSH [4], where the space and time complexities are $O(\sqrt{dk})$ and $O(d\sqrt{k})$, to generate binary codes of size k for a $\sqrt{d}$ by $\sqrt{d}$ matrix data. While promising results for hashing with bilinear projection are reported in [4], its theoretical analysis is not available yet.

In this paper we analyze a bilinear random projection method where feature matrices are transformed to binary codes by two smaller random projection matrices. We base our theoretical analysis on extending LSH from shift-invariant kernels (referred to as LSH-SIK [5]). We consider a hash function $h(\cdot) : \mathbb{R}^{d_w \times d_v} \mapsto \{0,1\}$ that is of the form

$$h(X) \triangleq \frac{1}{2}\Big\{1 + \mathrm{sgn}\Big(\cos\big(w^\top X v + b\big) + t\Big)\Big\}, \quad (1)$$

where $w \sim \mathcal{N}(0, I)$, $b \sim \mathrm{Unif}[0, 2\pi]$, and $t \sim \mathrm{Unif}[-1, 1]$. To produce binary code of size $k = k_w k_v$, the hash function $H(\cdot) : \mathbb{R}^{d_w \times d_v} \mapsto \{0,1\}^k$ takes the form:

$$H(X) \triangleq \frac{1}{2}\Big\{1 + \mathrm{sgn}\Big(\cos\big(\mathrm{vec}(W^\top X V) + b\big) + t\Big)\Big\}, \quad (2)$$

where each column of W or of V is independently drawn from spherical Gaussian with zero mean and unit variance, each entry of $b \in \mathbb{R}^k$ or of $t \in \mathbb{R}^k$ is drawn uniformly from $[0, 2\pi]$ and $[-1, 1]$, respectively.

We attempt to answer two questions on whether: (1) bilinear random projections also yield similarity-preserving binary codes like the original LSH-SIK; (2) there is performance gain or degradation when bilinear random projections are adopted instead of a large linear projection.

Regarding the first question, we present upper and lower bounds on the expected Hamming distance between binary codes produced by bilinear random projections. Theorem 1 and figure 1 (a) show that bilinear projections can generate similarity-preserving binary codes, where the expected Hamming distance is upper and lower bounded in terms of $\kappa_g\big(\mathrm{vec}(X - Y)\big)$.

Theorem 1. *Define the functions*

$$g_1(\zeta) \triangleq \frac{4}{\pi^2}\left(1 - \zeta^{0.79}\right),$$
$$g_2(\zeta) \triangleq \min\left\{\frac{1}{2}\sqrt{1-\zeta}, \frac{4}{\pi^2}\left(1 - \frac{2}{3}\zeta\right)\right\},$$

where $\zeta \in [0,1]$ and $g_1(0) = g_2(0) = \frac{4}{\pi^2}$, $g_1(1) = g_2(1) = 0$. Gaussian kernel κ_g is shift-invariant, normalized, and satisfies $\kappa_g(\alpha x - \alpha y) \le \kappa_g(x - y)$ for any $\alpha \ge 1$. Then the expected Hamming distance between any two embedded points computed by bilinear LSH-SIK satisfies

$$g_1\Big(\kappa_g(\tau)\Big) \le \mathbb{E}\Big[\mathcal{I}\big[h(X) \neq h(Y)\big]\Big] \le g_2\Big(\kappa_g(\tau)\Big), \quad (3)$$

where $\tau = vec(X - Y)$.

This is an extended abstract. The full paper is available at the Computer Vision Foundation webpage.

Regarding the second question, we analyze the upper and lower bounds on covariance between two bits of binary codes, showing that the correlation between two bits is small. Theorem 2 and figure 1 (b) show that the upper bound on covariance between two bits induced by bilinear projections is small, establishing the reason why bilinear projections performs well enough in case of a large number of bits. We can easily see that the covariance between the two bits for the highly similar ($\kappa_g(\mathrm{vec}(X - Y)) \approx 1$) is nearly zero, indicating that there is no correlation between the two bits.

Theorem 2. *Given the hash functions $h_i(\cdot)$ and $h_j(\cdot)$ which share one of projection vectors, the upper bound on the covariance between the two bits is derived as*

$$cov(\cdot) \le \frac{64}{\pi^4}\left\{\left(\sum_{m=1}^{\infty}\frac{\kappa_g\big(vec(X-Y)\big)^{0.79m^2}}{4m^2-1}\right)^2 - \left(\sum_{m=1}^{\infty}\frac{\kappa_g\big(vec(X-Y)\big)^{m^2}}{4m^2-1}\right)^2\right\},$$

where $\kappa_g(\cdot)$ is the Gaussian kernel and $cov(\cdot)$ is the covariance between two bits defined as

$$cov(\cdot) = \mathbb{E}\Big[\mathcal{I}\big[h_i(X) \neq h_i(Y)\big]\mathcal{I}\big[h_j(X) \neq h_j(Y)\big]\Big] - \mathbb{E}\Big[\mathcal{I}\big[h_i(X) \neq h_i(Y)\big]\Big]\mathbb{E}\Big[\mathcal{I}\big[h_j(X) \neq h_j(Y)\big]\Big].$$

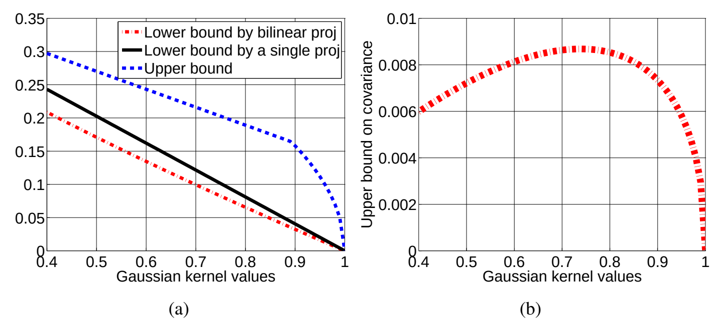

Figure 1: Left panel (a) presents that upper and lower bounds on the expected Hamming distance between binary codes computed by bilinear projections and a single projection. Right panel (b) shows that upper bound on covariance between the two bits induced by bilinear projections.

To conclude, our theoretical analysis have confirmed that: (1) randomized bilinear projection yields similarity-preserving binary codes in the sense of Gaussian kernel; (2) the performance of LSH-SIK with bilinear projections is comparable to LSH-SIK with a single large projection.

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