# Structured Sparse Subspace Clustering: A Unified Optimization Framework

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In many real-world applications, we need to deal with high-dimensional datasets, such as images, videos, text, and more. In practice, such high-dimensional datasets can be well approximated by multiple low-dimensional subspaces corresponding to multiple classes or categories. For example, the feature point trajectories associated with a rigidly moving object in a video lie in an affine subspace (of dimension up to 4), and face images of a subject under varying illumination lie in a linear subspace (of dimension up to 9). Therefore, the task, known in the literature as *subspace clustering* [6], is to segment the data into the corresponding subspaces and finds multiple applications in computer vision.

State of the art approaches [1, 2, 3, 4, 5, 7] for solving this problem follow a two-stage approach: a) Construct an affinity matrix between points by exploiting the 'self-expressiveness' property of the data, which allows any data point to be represented as a linear (or affine) combination of the other data points; b) Apply spectral clustering on the affinity matrix to recover the data segmentation. Dividing the problem in two steps is, on the one hand, appealing because the first step can be solved using convex optimization techniques, while the second one can be solved using existing spectral clustering techniques. On the other hand, its major disadvantage is that the natural relationship between the affinity matrix and the segmentation of the data is not explicitly captured.

In this paper, we attempt to integrate the two separate stages into one unified optimization framework. One important motivating observation is that a perfect subspace clustering can often be obtained from an imperfect affinity matrix. In other words, the spectral clustering step can clean up the disturbance in the affinity matrix – which can be viewed as a process of information gain by denoising. Because of this, if we feed back the information gain properly, it may help the self-expressiveness model to yield a better affinity matrix.

To jointly estimate the clustering and affinity matrix, we define a *subspace structured* $\ell_1$ norm as follows:

$$\|Z\|_{1,Q} \doteq \|(\mathbf{1}\mathbf{1}^\top + \alpha\Theta)\odot Z\|_1 \tag{1}$$

where $\alpha > 0$ is a tradeoff parameter, $\Theta_{ij} \in \{0,1\}$ indicates whether two data points belong to the same subspace in which $\Theta_{ij} = 0$ if point $i$ and $j$ lie in the same subspace and otherwise $\Theta_{ij} = 1$, and $\mathbf{1}$ is the vector of all ones of appropriate dimension.

Equipped with the *subspace structured* $\ell_1$ norm of $Z$, we then define the unified optimization framework for subspace clustering as follows:

$$\min_{Z,E,Q} \|Z\|_{1,Q} + \lambda\|E\|_\ell \;\text{ s.t. }\; X = XZ+E,\; \text{diag}(Z) = \mathbf{0},\; Q \in \mathcal{Q}, \tag{2}$$

where $\mathcal{Q}$ is the set of all valid binary segmentation matrices defined as

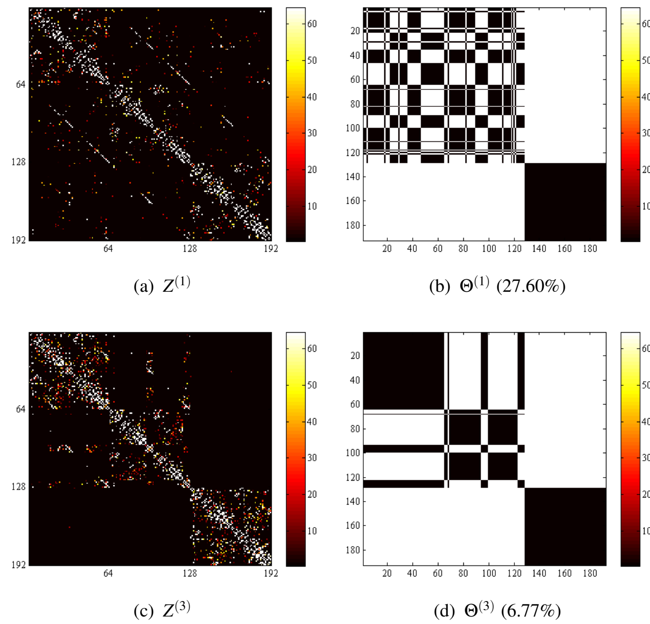
$$\mathcal{Q} = \{Q \in \{0,1\}^{N\times k} : Q\mathbf{1} = \mathbf{1} \text{ and } \text{rank}(Q) = k\}, \tag{3}$$

and the norm $\|\cdot\|_\ell$ on the error term $E$ depends upon the prior knowledge about the pattern of noise or corruptions. We call problem (2) *Structured Sparse Subspace Clustering* (SSSC or $\mathbf{S^3}$C).

The solution to the optimization problem in (2) is based on solving the following two subproblems alternatively: a) Find $Z$ and $E$ given $Q$ by solving a weighted sparse representation problem; b) Find $Q$ given $Z$ and $E$ by spectral clustering. We solve this problem efficiently via a combination of an alternating direction method of multipliers with spectral clustering. Experiments on a synthetic data, the Hopkins 155 motion segmentation database, and the Extended Yale B data set demonstrate its effectiveness.

Some results are presented in Figure 1, Table 1 and 2. Figure 1 shows the improvement in both the affinity matrix and the subspace clustering using $\mathbf{S^3}$C over SSC on a subset of face images of three subjects from the Extended Yale B data set. Table 1 shows the results on a synthetic data set which consists of 150 data points sampled from 15 linear subspaces of dimension 5 in 100-dimensional space. Table 2 shows the results on the Extended Yale B data set. As can be observed, our method consistently outperforms SSC on both synthetic and real world data set.

This is an extended abstract. The full paper is available at the Computer Vision Foundation webpage.



(a) $Z^{(1)}$ (b) $\Theta^{(1)}$ (27.60%) (c) $Z^{(3)}$ (d) $\Theta^{(3)}$ (6.77%)

Figure 1: Visualization of matrices $Z^{(t)}$ and $\Theta^{(t)}$ in the first and the third iterations of $\mathbf{S^3}$C algorithm. Note that the first iteration of $\mathbf{S^3}$C is effectively a SSC and hence the images of $Z^{(1)}$ and $\Theta^{(1)}$ in panel (a) and (b) are the representation matrix and the structure matrix of SSC. The images in panel (c) and (d) are the representation matrix $Z^{(t)}$ and structure matrix $\Theta^{(t)}$ of $\mathbf{S^3}$C when converged ($t = 3$). The percentage numbers in bracket are the corresponding clustering errors.

| Corruptions (%) | 0 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 |
|---|---|---|---|---|---|---|---|---|---|
| SSC | 1.43 | 1.93 | 2.17 | 4.27 | 16.87 | 32.50 | 54.47 | 62.43 | 68.87 |
| **Our S$^3$C** | **0.30** | **0.33** | **0.90** | **2.97** | **10.70** | **23.67** | **50.50** | **60.70** | **67.97** |

Table 1: Clustering Errors on Synthetic Data Set. The best results are in bold font.

| no. subj. | 2 | | 3 | | 5 | | 8 | | 10 | |
|---|---|---|---|---|---|---|---|---|---|---|
| ERR (%) | Mean | Med | Mean | Med | Mean | Med | Mean | Med | Mean | Med |
| SSC [1] | 1.87 | **0.00** | 3.35 | 0.78 | 4.32 | 2.81 | 5.99 | 4.49 | 7.29 | 5.47 |
| **S$^3$C** | **1.27** | **0.00** | **2.71** | **0.52** | **3.41** | **1.25** | **4.15** | **2.93** | **5.16** | **4.22** |

Table 2: Clustering Errors on Extended Yale B Data Set. The best results are in bold font.

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