Efficient Parallel Optimization for Potts Energy with Hierarchical Fusion

Olga Veksler
Department of Computer Science, University of Western Ontario.

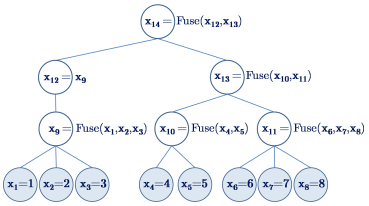

Figure 1: Illustration of hierarchical fusion.

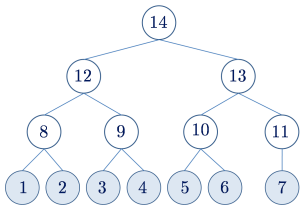

Figure 2: Our label tree structure.

Energies with Potts pairwise term are frequently used in various vision and graphics applications [1]. Potts model is popular because it is, arguably, the simplest useful model that imposes smoothness on the solution. Although optimizing Potts energy is NP-hard [1], there are efficient approximate optimization algorithms [1, 3].

We propose a graph-cut approach to Potts energy minimization that is easy to parallelize, see 1. Our approach is based on extending the hierarchical fusion (h-fusion) algorithm of [2], which is based on the fusion algorithm of [5]. The h-fusion algorithm of [2] is almost identical to the hierarchical algorithm of [4] in the absence of the label cost terms.

In [5] they also propose a graph-cut based parallel optimization algorithm. However, their approach, in general, has no optimality guarantees. Our approach has optimality factor $O(\log_2 k)$ even after just one iteration over all labels, i.e. after solving $k-1$ max flow problems, where k is the number of labels. Intuitively, it seems plausible that one has to solve at least that number of max-flow problems to have any optimality guarantees.

Our approximation factor this is not as good as the factor of 2 of the expansion algorithm. However in practice we achieve very good results. In particular, the results of our algorithm after one iteration are always better than the results after one iteration of expansion, see Fig. 3. In fact, we show that expansion algorithm has an approximation factor of $O(k)$ after one iteration. This factor is much worse than that of our algorithm, which helps to explain why we get better results after one iteration. For some applications, one iteration may be sufficient, see Fig. 4. Also, if one has a time-critical application, one iteration may be all that one can afford.

When Potts model is used for pairwise terms, the energy is:

$$f(x) = \sum_{p \in P} f_p(x_p) + \sum_{\{p,q\} \in E} w_{pq} \cdot [x_p \neq x_q], \qquad (1)$$

where P is the set of image pixels, E is the set of neighboring pixel pairs, $[\cdot]$ is the Iverson bracket, and w_{pq} is a non-negative coefficient.

Theorem 1. *Let f_{pq} be Potts pairwise term. Then the solution found with Alg. 1 is within a factor of $O(\log k)$ from the global optimum.*

Our Alg. 1, based on the original hierarchical fusion algorithm in [2, 4] is not iterative. It is not immediately obvious how to make it iterative, since it does not even make use of any initial solution, unlike most iterative algorithms. To make it iterative, we make the following observation. Let $\hat{x}$ be the solution obtained by Alg. 1. Let

$$E' = \{\{p,q\} \in E \,|\, \hat{x}_p \neq \hat{x}_q\}$$

We can execute Alg. 1 for a new energy function g, which is defined as follows. The unary terms are left unchanged: $g_p(x_p) = f_p(x_p)$, and the pairwise terms are changed as:

$$g_{pq}(x_p, x_q) = \begin{cases} f_{pq}(x_p, x_q) & \text{for } \{p,q\} \in E \setminus E' \\ 0 & \text{for } \{p,q\} \in E' \end{cases} \qquad (2)$$

Proposition 1. *Let $x^f = \hat{x}$ the the labeling Alg. 1 returns when optimizing $f(x)$. Let $g(x)$ be as defined in Eq. 2, and let x^g, be the labelling Alg. 1 returns when optimizing $g(x)$. Then $f(x^g) \leq f(x^f)$.*

Thus Alg. 1 can be iterated until convergence by constructing new set E' after each iteration.

Algorithm 1: OUR HIERARCHICAL FUSION

```
for each i = 1,...,k do
    for each p ∈ P do
        x_p^i = i
for each i = 1,...,k do
    ready(i) := true
for each i = k+1,...,2k do
    ready(i) := false
// perform fusions
for each j = k+1, k+2, ..., r in parallel do
    a := C_l(j)
    b := C_r(j)
    if ready(a) = true and ready(b) = true
        x^j := argmin_{x ∈ Fuse(x^a, x^b)} f(x)
        ready(j)= true
return x^r
```

Image	k	expansion energy	expansion time	h-fusion energy	h-fusion time
tsukuba	16	1074813	0.552	1068527	0.294
venus	20	2329733	1.227	2319501	0.593
teddy	60	3961242	3.247	3906396	1.32
cones	60	4624147	3.207	4586747	1.216
sawtooth	20	2740786	0.987	2730520	0.499
bull	20	2028894	0.969	2018992	0.505
barn1	20	2590260	0.898	2582261	0.482

Figure 3: Result comparison after one iteration of the expansion algorithm vs. one iteration of our approach (h-fusion). Running times are in seconds.

[1] Yuri Boykov, Olga Veksler, and Ramin Zabih. Fast Approximate Energy Minimization via Graph Cuts. *IEEE Transactions on Pattern Recognition and Machine Intelligence (TPAMI)*, 23(11):1222–1239, 2001.

[2] Andrew Delong, Lena Gorelick, Olga Veksler, and Yuri Boykov. Minimizing energies with hierarchical costs. *International Journal of Computer Vision*, 100(1):38–58, 2012.

[3] Vladimir Kolmogorov. Convergent Tree-Reweighted Message Passing for Energy Minimization. *IEEE Transactions on Pattern Analysis and Machine Intelligence (TPAMI)*, 28(10):1568–1583, October 2006.

[4] M. Pawan Kumar and Daphne Koller. MAP estimation of semi-metric MRFs via hierarchical graph cuts. In *Conference on Uncertainty in Artificial Intelligence*, pages 313–320, June 2009.

[5] Victor Lempitsky, Carsten Rother, Stephan Roth, and Andrew Blake. Fusion moves for markov random field optimization. *IEEE Transactions on Pattern Analysis and Machine Inference (TPAMI)*, 32:1392–1405, August 2010.

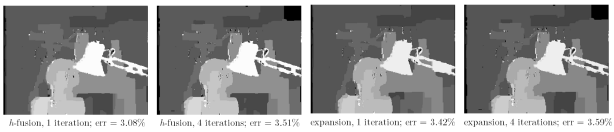

Figure 4: Disparity maps for *tsukuba* scene.

This is an extended abstract. The full paper is available at the Computer Vision Foundation webpage.