Homography Estimation from the Common Self-polar Triangle of Separate Ellipses

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Abstract

How to avoid ambiguity is a challenging problem for conic-based homography estimation. In this paper, we address the problem of homography estimation from two separate ellipses. We find that any two ellipses have a unique common self-polar triangle, which can provide three line correspondences. Furthermore, by investigating the location features of the common self-polar triangle, we show that one vertex of the triangle lies outside of both ellipses, while the other two vertices lies inside the ellipses separately. Accordingly, one more line correspondence can be obtained from the intersections of the conics and the common self-polar triangle. Therefore, four line correspondences can be obtained based on the common self-polar triangle, which can provide enough constraints for the homography estimation. The main contributions in this paper include: (1) A new discovery on the location features of the common self-polar triangle of separate ellipses. (2) A novel approach for homography estimation. Simulate experiments and real experiments are conducted to demonstrate the feasibility and accuracy of our approach.

1. Introduction

In computer vision, homograhy estimation arises in many situations [6], such as camera calibration [26], 3D reconstruction [2, 8], visual metrology [15], stereo vision [16] and scene understanding [1, 17]. The definition of homograhy described in [10] is that a homography is an invertible mapping from $\mathbf{P}^2$ to itself such that three points lie on the same line if and only if their mapped points are also collinear. The purpose of homography estimation is to find the mapping matrix. In order to recover the matrix, object correspondences are needed. Usually, there are three popular types of objects used in homography estimation, which are points, lines and conics [6]. In this paper, we are interested in conics. Conics are widely used in computer vision [14, 18] due to two reasons: (1) They are well studied in mathematics and can be represented by a simple matrix. (2) They can be detected and estimated robustly by existing mature algorithms. In order to estimate homography from conics, Sugimoto presented a direct conic based homography estimation method in [21], but it requires seven conic correspondences and it has to solve the problem of ambiguity by conducting back projection. Later, by considering two conics together, Kannala et al. [13] presented an algorithm for the minimal case of a pair of conic correspondences, which leads to 4 solutions. In [19], Rothwell et al. showed a straightforward approach based on the fact that any two conics always intersect in 4 points (real or complex). Unfortunately, their method has 24 solutions and involves in solving quartic equations. Especially, they addressed the difficulty in matching points from two separate ellipses. They pointed out that there is no way to order the complex points. Even the intersections pairs are complex conjugate pairs, there are still 8 combinations. On this matter, one might use quasi-affine invariant [25] or use absolute signature and limit points [9] to distinguish the two complex conjugate pairs. However, It still results in 4 combinations. The reason is that for each pair of complex conjugate points, there are two combinations. In [4], Chum and Matas found the homograhpy by first order Taylor expansions at two (or more) points from two or more correspondences of local elliptical features. However, the prerequisite of their method is that the metric information of at least two point correspondences should be known. In [5], Conomis recovered the homography by using the pole-polar relationship. Their approach first finds the correspondences using the common poles. Further points are computed by intersecting the polar of the pole with the conic. However, point order is not preserved under pojective transformation. To solve this prob-

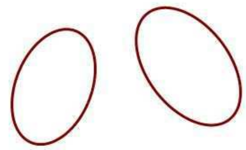

Figure 1. Two separate ellipses.

lem, an affine ordering strategy is proposed, but Chum and Matas [4] pointed out the strategy does not often provide correct correspondence of the poles, which leads to a potentially high number of possible homgoraphy models. In [24], Wright et al. recovered the homography using four bitangent lines of two ellipses. The problem is the same as using four intersection points. To avid ambiguity, lots of assumptions should be made in their method.

In this paper, we focus on two separate ellipses (see Fig.1). We find that homography estimation from separate conics has real application. For instance, in [24], Wright et al. used it in forensic blood splatter reconstruction. We try to solve this problem using the common self-polar triangle of conics. Previously, the common self-polar triangle has been used in camera calibration [11, 12] and position relationship discussion [22]. So far, to the best of our knowledge, there are no studies on the location features of the common self-polar triangle of separate ellipses. We find that any two separate ellipses have a unique common self-polar triangle, which can provide three line correspondences. Furthermore, by investigating the location features of the common self-polar triangle, we show that one more line correspondence can be obtained from the intersections of the conics and the common self-polar triangle. In total, four line correspondences can be obtained based on the common self-polar triangle, which can provide enough constraints for the homography estimation. Our approach has three advantages: (1) No requirements on the physical information of the patterns and the camera. (2) All computations involved are linear. (3) No ambiguity on the solution. To evaluate our approach, we conduct simulate experiments and real experiments, whereby accurate results are achieved.

Note that our approach is different from [5], even we both use the pole-polar relationship. Paper [5] has not explored the location features of the common poles and they need to carefully order the points when they try to find more point correspondences by intersecting the polar with the conics. In our approach, we explore the location feature of the common self-polar triangle, and four line correspondences can be determined easily, efficiently and without any ambiguity. Our contribution in this paper includes: (1) A new discovery on the location features of the common self-polar triangle of separate ellipses. (2) A novel approach for homography estimation.

The remainder of this paper is organized as follows. Section 2 briefly introduces related notations and theorems. Section 3 discusses the location features of the common self-polar triangle of separate ellipses. Section 4 describes homography estimation method. Section 5 shows the experimental results on synthetic and real data sets. Finally, the concluding remarks are drawn in Section 6.

2. Preliminaries

2.1. Homography

Let $(\mathbf{x}, \mathbf{x}^{'})$ be a point correspondence from two images of the same scene plane. In the homogenous coordinate system, the homography between these two images can be expressed as

$$s\,[x^{'}\;\, y^{'}\;\, 1]^{\mathbf{T}} = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix} [x\; y\; 1]^{\mathbf{T}} \tag{1}$$

or

$$s\mathbf{x}^{'} = \mathbf{H}\mathbf{x}, \tag{2}$$

where, $\mathbf{H}$ is usually a non-singular 3 by 3 homogenous matrix. $\mathbf{H}$ has 8 degrees of freedom, which can be estimated from four point correspondences(no three points are collinear).

Let line $\mathbf{l}$ go through $\mathbf{x}$ in the first image and line $\mathbf{l}^{'}$ go through $\mathbf{x}^{'}$ in the second image. Due to the duality principle [10] about points and lines, the mapping process can be expressed as

$$s\mathbf{l} = \mathbf{H}^{\mathbf{T}}\mathbf{l}^{'}. \tag{3}$$

Similarly, at least four line correspondences are needed to estimate the homography. In this paper, we will use the line correspondences.

Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be the images of conic $\mathbf{C}$ under two different views. The projective transformations are $\mathbf{H}_1$ and $\mathbf{H}_2$ respectively. The imaged conics $\mathbf{C}_1$ and $\mathbf{C}_2$ can be expressed as

$$\begin{aligned} \mathbf{C}_1 &= \mathbf{H_1}^{-\mathbf{T}}\mathbf{C}\mathbf{H_1}^{-\mathbf{1}} \\ \mathbf{C}_2 &= \mathbf{H_2}^{-\mathbf{T}}\mathbf{C}\mathbf{H_2}^{-\mathbf{1}}. \end{aligned} \tag{4}$$

Furthermore, we can obtain

$$\mathbf{C}_2 = (\mathbf{H_2}\mathbf{H_1}^{-1})^{-\mathbf{T}}\mathbf{C_1}(\mathbf{H_2}\mathbf{H_1}^{-1})^{-\mathbf{1}}. \tag{5}$$

Thus, the homography from the first image to second image is $\mathbf{H_2}\mathbf{H_1}^{-1}$ [10].

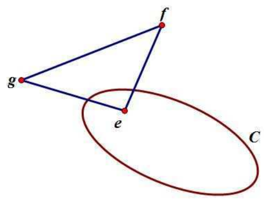

Figure 2. $\triangle \mathbf{efg}$ is a self-polar triangle with respect to conic $\mathbf{C}$ when polars of e, f and g are lines fg, eg and ef, respectively.

2.2. Pole-polar Relationship and Self-polar Triangle

A point $\mathbf{x}$ and conic $\mathbf{C}$ define a line $\mathbf{l} = \mathbf{Cx}$. The line $\mathbf{l}$ is called the polar of $\mathbf{x}$ with respect to $\mathbf{C}$, and the point $\mathbf{x}$ is the pole of $\mathbf{l}$ with respect to $\mathbf{C}$.

If the poles of a conic form the vertices of a triangle and their respective polars form its opposite sides, it is called a self-polar triangle (see Fig.2). If a self-polar triangle is common to two conics, it is called common self-polar triangle (see Fig.6) [23].

In this paper, we will use two results from projective geometry [10, 7, 20] They are:

Result 1. If the pole lies on the conic, the polar is the poles tangent line; If the pole lies outside the conic, the polar intersects the conic in two points; If the pole lies inside the conic, the polar has no real intersection points with the conic.

Result 2. If two conics intersect in four distinct points, they have one and only one common self-polar triangle.

3. Location Features of the Common Self-polar Triangle of Separate Ellipses

Before exploring the location features of the common self-polar triangle of separate ellipses, one proposition is established below.

Proposition 1. *Two separate ellipses have a unique common self-polar triangle.*

Proof. Considering intersection points of two separate ellipses, it is easy to find that they have four imaginary intersection points, which fall into two conjugate pairs. Obviously, these four intersection points are distinct. According to the Result 2 in Section 2.2, we can conclude that two separate ellipses have a unique common self-polar triangle.

3.1. The Common Self-polar Triangle Location

Based on the fact that two separate ellipses have a unique common self-polar triangle, we investigate the location features and following results are achieved.

Property. *Let two separate ellipses be $\mathbf{C}_1$ and $\mathbf{C}_2$, and let points $\mathbf{e}$,$\mathbf{f}$ and $\mathbf{g}$ be the vertices of their common self-polar triangle. Then one vertex lies outside of both $\mathbf{C}_1$ and $\mathbf{C}_2$, and the other two vertices lies inside $\mathbf{C}_1$ and $\mathbf{C}_2$ separately.*

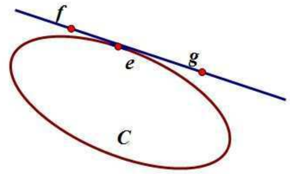

Figure 3. When the pole e is on the conic, the polar fg is the tangent line.

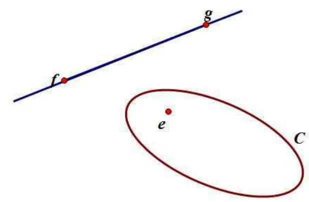

Figure 4. When the pole e lies inside the conic, the polar fg is outside the conic.

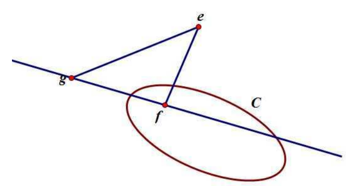

Figure 5. When the pole e lies outside the conic, there is one and only one point from points f and g lies inside the conic.

Proof. Let $\mathbf{C}$ be a conic and $\triangle efg$ be a self-polar triangle of $\mathbf{C}$. Here, we proof that there is one and only one vertex of $\triangle efg$ lies inside $\mathbf{C}$ (see Fig.2).

We readonly choose one vertex from the self polar triangle and let it be point $\mathbf{e}$. In terms of the definition of self-polar triangle in Section 2.2, we know that the opposite side of the pole is the polar, which means the polar of the pole $\mathbf{e}$ is the line going through point $\mathbf{f}$ and point $\mathbf{g}$. It is easy to know point $\mathbf{e}$ can not lie on the conic. If $\mathbf{e}$ lies on the conic, according to Result 1 in Section 2.2, the polar of $\mathbf{e}$ will go through $\mathbf{e}$. Then points $\mathbf{e}$, $\mathbf{f}$ and $\mathbf{g}$ lies on the same line (see Fig.3), which can not form a triangle.

Based on the above analysis, we know $\mathbf{e}$ either lies inside conic $\mathbf{C}$ or outside conic $\mathbf{C}$. If point $\mathbf{e}$ lies inside $\mathbf{C}$, according to Result 1 in Section 2.2, the polar of point $\mathbf{e}$ has no intersections with coinic $\mathbf{C}$. Consequently, we have that point $\mathbf{f}$ and point $\mathbf{g}$ lies outside $\mathbf{C}$ (see Fig.4).

If point $\mathbf{e}$ lies outside conic $\mathbf{C}$, the polar of point $\mathbf{e}$ has two intersections with conic $\mathbf{C}$. Then there are two cases: (1) Points $\mathbf{f}$ and $\mathbf{g}$ are both inside the conic. (2) One point from points $\mathbf{f}$ and $\mathbf{g}$ lies inside the conic.

Let us consider the first case. If point $\mathbf{f}$ lies inside the

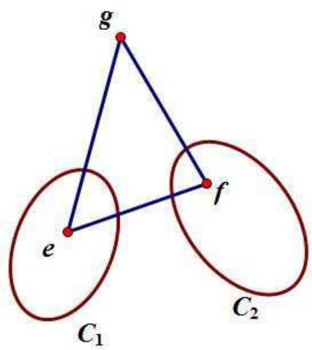

Figure 6. $\triangle\mathbf{efg}$ is the common self-polar triangle with respect to conic $\mathbf{C}_1$ and $\mathbf{C}_2$.

conic, according to Result 1 in Section 2.2, the line (polar) going through points $\mathbf{e}$ and $\mathbf{g}$ has no intersections with C, which implies point $\mathbf{g}$ is outside the conic. Obviously, the first case is not true. Then, it must be the second case (see Fig.5).

Finally, we can conclude that for any self-polar triangle of a conic, there is one and only one vertex of this triangle lies inside the conic. Since $\triangle efg$ is the common self-polar triangle of $\mathbf{C}_1$ and $\mathbf{C}_2$, Then $\triangle efg$ has one and only one vertex lies inside $\mathbf{C}_1$, and has one and only one vertex lies inside conic $\mathbf{C}_2$. The third vertex lies outside of both $\mathbf{C}_1$ and $\mathbf{C}_2$ (see Fig.6). □

This Property is very useful. In Section 4, we will derive one important line correspondence from it without any ambiguity.

3.2. The Common self-polar Triangle Recovery

Let point $\mathbf{x}$ and line $\mathbf{l}$ are the common pole-polar of $\mathbf{C}_1$ and $\mathbf{C}_2$. The following relationship should be satisfied:

$$\begin{aligned} x &= \mathbf{C}_1{}^{-1}\mathbf{l} \\ x &= \lambda\mathbf{C}_2{}^{-1}\mathbf{l}, \end{aligned} \tag{6}$$

where λ is a scalar parameter. Subtracting the equations in (6), we get $(\mathbf{C}_1{}^{-1} - \lambda\mathbf{C}_2{}^{-1})\mathbf{l} = 0$. By multiplying $\mathbf{C}_2$ on both sides, we obtain the following equation:

$$(\mathbf{C}_2\mathbf{C}_1{}^{-1} - \lambda\mathbf{I})\mathbf{l} \quad = \quad 0. \tag{7}$$

From the equation of (7), we find the common polars for $\mathbf{C}_1$ and $\mathbf{C}_2$ are the eigenvectors of $\mathbf{C}_2\mathbf{C}_1{}^{-1}$. Since we have proved that two separate ellipses have a unique self-polar triangle, the eigenvalues should be distinct.

Similarly, we can find the vertices of the common self-polar triangle by computing the eigenvectors of $\mathbf{C}_2{}^{-1}\mathbf{C}_1$.

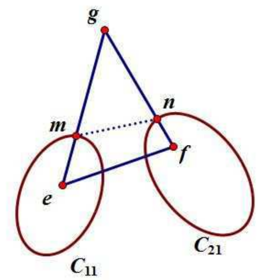

Figure 7. When $\triangle efg$ is the common self-polar triangle of $\mathbf{C}_{11}$, $\mathbf{C}_{21}$, one more line mn within the triangle can be uniquely determined by the intersections.

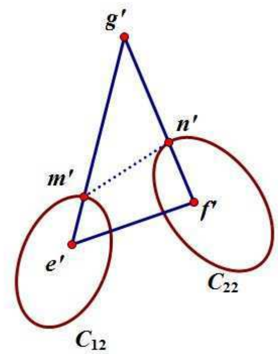

Figure 8. Line $m^{'}n^{'}$ is the correspond line of mn in Fig.7.

4. Homography Estimation

4.1. Line Correspondences from the Common self-polar Triangle

Let the first image of two separate ellipses $\mathbf{C}_1$ and $\mathbf{C}_2$ be $\mathbf{C}_{11}$ and $\mathbf{C}_{21}$. Let the second image be $\mathbf{C}_{12}$ and $\mathbf{C}_{22}$. Let the transformation from the first image to the second image is $\mathbf{H}$. We have:

$$\begin{aligned} \mathbf{C}_{12} &= \mathbf{H^{-T}}\mathbf{C}_{11}\mathbf{H^{-1}} \\ \mathbf{C}_{22} &= \mathbf{H^{-T}}\mathbf{C}_{21}\mathbf{H^{-1}}. \end{aligned} \tag{8}$$

By computing the product $\mathbf{C}_{22}\mathbf{C}_{12}{}^{\mathbf{-1}}$, we obtain:

$$\begin{aligned} \mathbf{C}_{22}\mathbf{C}_{12}{}^{\mathbf{-1}} &= (\mathbf{H^{-T}}\mathbf{C}_{21}\mathbf{H}^{-1})(\mathbf{H^{-T}}\mathbf{C}_{11}\mathbf{H}^{-1})^{-1} \\ &= \mathbf{H^{-T}}(\mathbf{C}_{21}\mathbf{C}_{11}{}^{-1})\mathbf{H^{T}}. \end{aligned} \tag{9}$$

Then,

$$\mathbf{C}_{21}\mathbf{C}_{11}{}^{\mathbf{-1}} = \mathbf{H^{T}}(\mathbf{C}_{22}\mathbf{C}_{12}{}^{-1})\mathbf{H^{-T}}. \tag{10}$$

We find that $\mathbf{C}_{21}\mathbf{C}_{11}{}^{-1}$ is similar to $\mathbf{C}_{22}\mathbf{C}_{12}{}^{\mathbf{-1}}$. If λ and $\mathbf{l}^{'}$ are eigenpairs of $\mathbf{C}_{22}\mathbf{C}_{12}{}^{-1}$, according to the property of similarity transformation, λ and $\mathbf{H^{T}}\mathbf{l}^{'}$ are eigenpairs of $\mathbf{C}_{21}\mathbf{C}_{11}{}^{\mathbf{-1}}$. Based on this observation, we can find three lines correspondences by matching the eigenvalues.

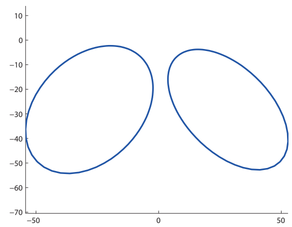

Figure 9. Two separate ellipses generated by computer.

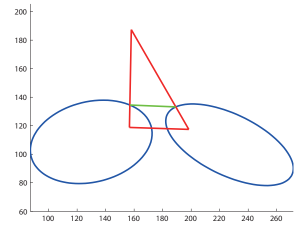

Figure 10. The first image with the common self-polar triangle.

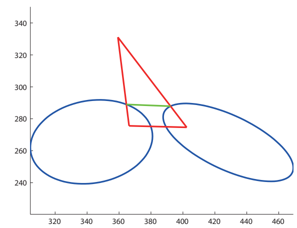

Figure 11. The second image with the common self-polar triangle.

4.2. One More Line Correspondence

As Section 2 discussed, at least 4 line correspondences are required to estimate the homography. Here, we demonstrate one more line correspondence can be determined without any ambiguity using the Property.

Let $\triangle efg$ be the common self-polar triangle of $\mathbf{C}_{11}$, $\mathbf{C}_{21}$ (see Fig.7). Let $\triangle e^{'}f^{'}g^{'}$ be the common self-polar triangle of $\mathbf{C}_{12}$, $\mathbf{C}_{22}$ (See Fig.8). According to the Property in Section 3, segment eg has one and only one intersection (point m) with $\mathbf{C}_{11}$, and segment fg has one and only one intersection (point n) with $\mathbf{C}_{21}$. As we know, collinearity is an invariant under projective transformation, so line mn in first image and line $m^{'}n^{'}$ (see Fig.8) are correspondence.

4.3. Homography Estimation Method

Based on the above analysis, the complete homography estimation algorithm consists of the following steps:
Step 1: Extract conic $\mathbf{C}_{11}$, $\mathbf{C}_{21}$, and $\mathbf{C}_{12}$, $\mathbf{C}_{22}$ from two images separately.
Step 2: Compute the common polars of $\mathbf{C}_{12}$ and $\mathbf{C}_{22}$. Let three eigenpairs be $(\lambda_1, \mathbf{l}_1^{'})$, $(\lambda_2, \mathbf{l}_2^{'})$, and $(\lambda_3, \mathbf{l}_3^{'})$.
Step 3: Compute $(\beta, \mathbf{l})$ of $\mathbf{C}_{11}$, $\mathbf{C}_{21}$. Let three eigenpairs be $(\beta_1, \mathbf{l}_1)$, $(\beta_2, \mathbf{l}_2)$, and $(\beta_3, \mathbf{l}_3)$.
Step 4: Find three line correspondences by matching the values of the λ, β.
Step 5: Find one more line correspondences by connecting intersections within the common self-polar triangle.
Step 6: Calculate homography matrix using existing algorithms in [10].

5. Experiments and Results

5.1. Synthetic Data

In the computer simulations, we first generate two separate ellipses (see Fig.9). Then we set two projective transformation matrix to obtain two images of the separate ellipses. We choose 100 points on each ellipse image, and Gaussian noise with zero-mean and σ standard deviation is added to these image points. Ellipses are fitted to these images using a least squares ellipse fitting algorithm, the common self-polar triangles are computed using the Equation (7) in Section 3 (see Fig.10 and Fig.11). We vary the noise level from 0 pixels to 0.6 pixels. For each noise level, we conduct 1000 independent trials, and the final results are shown in average (see Table 1). The ground truth (G.T.) is calculated using the rules in Section 2.1.

	G.T.	0	0.2	0.4	0.6
h_1	1.2690	1.2690	1.2842	1.2771	1.2795
h_2	0.3036	0.3036	0.3102	0.3145	0.3082
h_3	215.6545	215.6545	214.7491	215.2049	215.0966
h_4	0.1502	0.1502	0.1573	0.1545	0.1561
h_5	1.4101	1.4101	1.4209	1.4211	1.4161
h_6	147.9527	147.9527	147.2942	147.5542	147.5756
h_7	0.0005	0.0005	0.0005	0.0005	0.0005
h_8	0.0013	0.0013	0.0013	0.0013	0.0013
h_9	1.0000	1.0000	1.0000	1.0000	1.0000

Table 1. Homography results with 0, 0.2, 0.4 and 0.6 pixels noise

	0	0.2	0.4	0.6
e	0	0.1639	0.2756	0.7999
f	0	0.1729	0.1888	0.6566
g	0	0.3150	0.7704	1.9692

Table 2. Symmetric transfer error for three points with noise 0.2, 0.4 and 0.6 pixels.

From the results, we find that, when there is no noise, the results is the same as the ground truth. When add noise, errors for each parameters is very small.

In the second experiment, we just add noise on the second image. The first image has no noise. We treat the vertices $\mathbf{e}$, $\mathbf{f}$ and $\mathbf{g}$ of the common self-polar triangle as fixed points, then calculate the symmetric transfer error (STE) [10]. The distance value are shown in Table 2. From the results, we find errors are less than 2 pixels.

For both experiments, we also implement the algorithm in [5]. By carefully ordering the points, the same results

Figure 12. The first reference image: Two ellipses on one A4 paper.

Figure 13. The second reference image:Two oval shaped plates

are achieved. Therefore, we do not list the same data in the table.

5.2. Real Scene

We conduct two real experiments. In the first experiment, we print out two ellipses on one A4 paper. To make it not so ideal, we intentionally place something on the paper. Especially, the "Heart" shape is for the convenience to check the performance of our method. In the second experiment, we place two oval shaped plates on a desk. For these two experiments, real images are taken with a Nikon D300s camera. The image resolution is 2144×1424. The images are taken from different viewpoints and the focal lengths are different. The images of ellipses are extracted using Canny's edge detector [3], and ellipses are fitted to these images using a least squares ellipse fitting algorithm. One of the images is chosen as reference (see Fig.12 and Fig.13). We use our presented approach to compute the homography for each image. Finally, we warp the images back to the reference view. The results are listed in Figure 14 and Figure 15.

Note that the considered perspective projections in this paper are quasi-affine with respect to the ellipses and in one case, the homography results may not be accurate. When the scene contains two separate ellipses with a common axis of symmetry, the image plane should not be parallel to the scene when we capture an image, because they have one common pole at infinity.

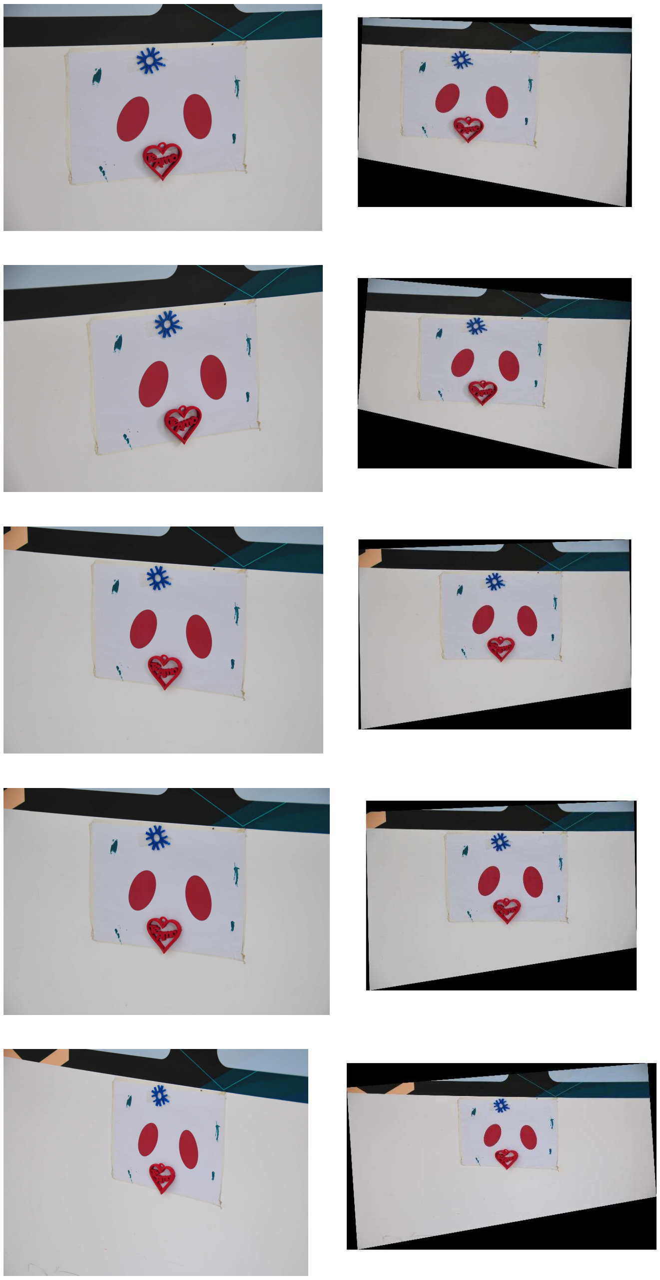

Figure 14. The left side are ellipse images taken from different viewpoints. The right side are images warped back to the reference view.

6. Concluding Remarks

We have investigated the location features of the common self-polar triangle of separate ellipses. Based on the common self-polar triangle, four line correspondences can be uniquely determined. Our approach is solidly derived from existing theories in projective geometry. All steps involved in our approach are linear and easy to implement.

Figure 15. The left side are plate images taken from different viewpoints. The right side are images warped back to the reference view.

We believe that our method can be easily extended to the case of more than two ellipses. In such case, any two separate ellipses can provide four line correspondences, if there are three or more ellipse correspondences, an overdetermined linear system can be obtained. Therefore, optimization methods can be used to obtain more accurate results. In future work, we will apply this theory into other distributions of two coplanar ellipses or circles.

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