Topological data analysis offers a rich source of valuable information to study vision problems. Yet, so far we lack a theoretically sound connection to popular kernel-based learning techniques, such as kernel SVMs or kernel PCA. In this work, we establish such a connection by designing a multi-scale kernel for persistence diagrams (see Fig. 1), a stable summary representation of topological features in data. We show that this kernel is positive definite and prove its stability with respect to the 1-Wasserstein distance. Experiments on two benchmark datasets for 3D shape classification/retrieval and texture recognition show considerable performance gains of the proposed method compared to an alternative approach that is based on the recently introduced persistence landscapes.

**Persistence diagrams.** Persistence diagrams are a concise description of the topological changes occurring in a growing sequence of shapes, called filtration. In particular, during the growth of a shape, holes of different dimension (i.e., gaps between components, tunnels, voids, etc.) may appear and disappear. Intuitively, a k-dimensional hole, born at time $b$ and filled at time $d$, gives rise to a point $(b, d)$ in the $k^{th}$ persistence diagram. A persistence diagram is thus a multiset of points in $\mathbb{R}^2$.

**Filtrations from functions.** A standard way of obtaining a filtration is to consider the sublevel sets $f^{-1}(-\infty, t]$ of a function $f : \Omega \to \mathbb{R}$ defined on some domain $\Omega$. For $t \in \mathbb{R}$, it is easy to see that the sublevel sets indeed form a filtration parameterized by $t$. We denote the resulting persistence diagram by $D_f$. Example(s): Consider a grayscale image, where $\Omega$ is the rectangular domain of the image and $f$ is the grayscale value at any point of the domain (i.e., at a particular pixel). A sublevel set would thus consist of all pixels of $\Omega$ with value up to a certain threshold $t$. Another example would be a piecewise linear function on a triangular mesh $\Omega$, such as the popular heat kernel signature [6]. Yet another commonly used filtration arises from point clouds $P$ embedded in $\mathbb{R}^n$, by considering the distance function $d_P(x) = \min_{p \in P} \|x - p\|$ on $\Omega = \mathbb{R}^n$. The sublevel sets of this function are unions of balls around $P$.

**The persistence scale-space (PSS) kernel.** We propose a stable multiscale kernel $k_{\sigma}$ for the set of persistence diagrams $D$. This kernel will be defined via a feature map $\Phi_{\sigma} : D \to L_2(\Omega)$, with $\Omega \subset \mathbb{R}^2$ denoting the closed half plane above the diagonal, i.e., $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 \geq x_1\}$.

Since a persistence diagram $D$ can be uniquely represented as a sum of Dirac delta distributions, we use the sum as an initial condition for a heat diffusion problem with a Dirichlet boundary condition on the diagonal. The solution of this partial differential equation (see paper) is an $L_2(\Omega)$ function for any chosen scale parameter $\sigma > 0$. We define the feature map (see Fig. 2 for an illustration) $\Phi_{\sigma} : D \to L_2(\Omega)$ at scale $\sigma > 0$ of a persistence diagram $D$ as $\Phi_{\sigma}(D) = \{u_{r,\sigma} : \Omega \times \mathbb{R}_+ \to \mathbb{R}\}$,

$$u(x,t) = \frac{1}{4\pi t} \sum_{p \in D} \exp \left(- \frac{\|x-p\|^2}{4t} \right) - \exp \left(- \frac{\|x-p\|^2}{4t} \right)$$

being the closed-form solution to the aforementioned partial differential equation. This map yields the persistence scale space kernel $k_{\sigma}$ on $D$ as

$$k_{\sigma}(F,G) = \langle \Phi_{\sigma}(F), \Phi_{\sigma}(G) \rangle_{L_2(\Omega)}$$

and we can derive a simple expression for evaluating the kernel:

$$k_{\sigma}(F,G) = \frac{1}{8\pi^2} \sum_{q \in G} \exp \left(- \frac{\|p-q\|^2}{8\sigma} \right) - \exp \left(- \frac{\|p-q\|^2}{8\sigma} \right)$$

where $q = (b, a)$ is $q = (a, b)$ mirrored at the diagonal.

**Evaluation.** In the paper, we report results on two vision tasks where persistent homology has already been shown to provide valuable discriminative information [3]: shape classification/retrieval (on SHREC 2014 [5]) and texture image classification (on the Outex_TC_00000 benchmark [4]); see Fig. 3 for an illustration of the datasets. We primarily compare against a kernel that can be constructed based on Bubenik’s concept of persistence landscapes [2], a representation of persistence diagrams as functions in the Banach space $L_p(\mathbb{R}^2)$. For $p = 2$, we can use the Hilbert space structure of $L_2(\mathbb{R}^2)$ to construct a kernel analogously to (2). Our experimental results are listed in the paper.

**Implementation.** DIPHA [1] is freely available at http://goo.gl/EXSpnl, the kernel implementation (compatible with DIPHA) will be made available right after the conference.